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The lift force on an arbitrarily shaped body in a steady incompressible inviscid linear shear flow with weak strain

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This paper presents a mathematical derivation using the classical theory of fluid dynamics for the force on an arbitrarily shaped body in a linear shear flow. To make the analysis tractable, the problem is linearized by assuming that the strain rate is weak and neglecting terms of the order of the strain rate squared. The argument generalizes previous established analytical results due to Darwin regarding the drift-volume and Lighthill for the asymptotic form of the rotational velocity field induced by the body. The final expression for the force is determined by generalizing an analytical argument due to Auton for the sphere. The results identify for the first time a rotational lift force component that occurs only when the body shape is truly asymmetric.

1. Introduction

The determination of the lift force on an arbitrarily shaped body in an inviscid incompressible fluid is of both fundamental and practical importance in fluid dynamics. In particular, the problem has high relevance to the study of bubble dynamics in turbulent flows with high Reynolds number. An air bubble in water, for example, experiences very little tangential stress thus making the free-slip boundary condition valid. The only analytic solution for three-dimensional bodies has been derived by Auton (1987) for the sphere, its symmetry providing simplifications in the analysis. In recognition of the continuing importance of the problem, Magnaudet & Legendre (1998) and Legendre & Magnaudet (1998) have made numerical calculations of the lift force on a spherical bubble for a range of Reynolds numbers and have investigated inviscid flow in the limit of large Reynolds number. The focus of this paper is on finding an analytical solution and, therefore, it is instructive to first discuss relevant theoretical studies. Together with the theoretical proof of Auton (1987) we must also consider the work on drift of both Darwin (1953) and Lighthill (1956, 1957). Drift, as discussed in § 1 of Lighthill, concerns the movement of material particles in steady uniform irrotational flow past bodies. Its relevance to this problem makes it the subject of more recent studies by Benjamin (1986), Eames, Belcher & Hunt (1994) and Yih (1985, 1995, 1997). In order to explain the relevance of drift to this study we shall introduce the concept of the local drift vector d_i corresponding to the irrotational velocity field v_i past the body.

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The ambient velocity field U_i for a uniform flow has the form

$$U_i = U\delta_{1i}. \tag{1.1}$$

Here, U is constant in space and time. Of its own accord, U_i would give rise to linear streamlines, as defined by

$$x_i(\boldsymbol{U}) = x_i^{-\infty} + Ut\delta_{1i} = x_1\delta_{1i} + x_{i\neq 1}^{-\infty}, \tag{1.2}$$

which we can imagine as starting at some far upstream position $x_i^{-\infty}$. Here, the superscript $-\infty$ indicates that the starting position is associated with a large negative value of x_1, namely $x_1 = x_1^{-\infty}$. The vector $x_{i\neq 1}^{-\infty} = (0, x_2^{-\infty}, x_3^{-\infty})$ denotes the finite off-axis coordinates of the starting position.

The irrotational velocity field v_i in the vicinity of the body, therefore, has the form

$$v_i = U\delta_{1i} + \Delta v_i. \tag{1.3}$$

Here, Δv_i is the irrotational disturbance velocity, so called because it is a perturbation to the uniform ambient velocity field caused by the body. Necessarily, therefore, v_i satisfies the normal velocity boundary condition $v_i\, n_i|_B = 0$ (here $|_B$ denotes evaluation on the surface $\mathscr{S}_B$ of the body). The streamlines corresponding to v_i are then defined by

$$x_i(\boldsymbol{v}) = x_i^{-\infty} + \int_{-\infty}^{t} v_i \,\mathrm{d}t. \tag{1.4}$$

Here, for consistency, time is defined as equal to $-\infty$ at the particle starting positions. The local drift, or drift vector d_i can now be defined as the relative displacement of fluid particles away from the positions they would have if moving with the ambient flow, thus

$$x_i(\boldsymbol{v}) = x_{i\neq 1}^{-\infty} + Ut\delta_{1i} - d_i. \tag{1.5}$$

Note that because the problem is steady, d_i is a function only of the space variable $\boldsymbol{x}$. It is, however, also considered here to be a function of time when viewed relative to a particle moving with the fluid. The position vector $\boldsymbol{x}$ in the fluid, therefore, also coincides with a fluid particle that started at position $\boldsymbol{x}^{-\infty}$ at time $t^{-\infty}$ and has reached the point $\boldsymbol{x}$ at time t. Thus, we can then write $\boldsymbol{x} = \boldsymbol{x}(t)$, but in recognition that the time t is strictly a function of $\boldsymbol{x}$. Darwin (1953) is particularly concerned with the limiting value of drift far downstream of the body. In particular $t^{+\infty}$ is a function of $x_2^{-\infty}$ and $x_3^{-\infty}$. Thus, if we adopt the superscript $+\infty$ to denote the far downstream, then taking the limit of (1.5), as $t \to t^{+\infty} = +\infty$, we obtain in (1.6) below Darwin's definition of total drift, here denoted D_i

$$x_i^{+\infty} = x_{i\neq 1}^{-\infty} + Ut^{+\infty}\delta_{1i} - D_i. \tag{1.6}$$

Importantly, in the case of the sphere, the symmetry of the flow results in the far downstream particles having the same off-axial displacements in the far downstream plane as they did at their starting positions. In our equation (1.6) this amounts to $x_{i\neq 1}^{+\infty} = x_{i\neq 1}^{-\infty}$ from which it follows that $D_{i\neq 1} = 0$ and, therefore, the only non-zero total drift component for the sphere is the axial component D_1. One implication of the body having arbitrary shape is that the off-axial total drift components $D_{i\neq 1}$ are non-zero and, therefore, this paper must address the application of Darwin's work to this situation.

Lighthill (1956) explores the interrelationship between drift and the rotational disturbance velocity field Δw_i generated by a body in a steady uniform shear flow whose ambient velocity has the form

$$U_i = (U - \Omega x_2)\,\delta_{1i}. \tag{1.7}$$

The corresponding ambient vorticity field Ω_i is then equal to

$$\Omega_i = \Omega\delta_{3i}. \tag{1.8}$$

In this case, the body gives rise to the irrotational disturbance velocity Δv_i defined above as well as a disturbance vorticity field $\Delta\omega_i$ whose associated rotational disturbance velocity is Δw_i. Thus, the total velocity field u_i is given by

$$u_i = v_i + w_i = (U\delta_{1i} + \Delta v_i) + (-\Omega x_2\delta_{1i} + \Delta w_i). \tag{1.9}$$

The rotational velocity w_i must, therefore, satisfy the normal boundary condition $w_i n_i|_B = 0$ on $\mathscr{S}_B$. As explained at the beginning of §3 of Lighthill (1956, p. 36), Δw_i must equal the sum of the Biot-Savart integral Δw_i^{BS} and a corresponding irrotational velocity Δv_i^{Ω}, the latter being required to satisfy the velocity boundary condition. Thus

$$w_i = -\Omega x_2\delta_{1i} + \Delta w_i^{BS}(\boldsymbol{u}) + \Delta v_i^{\Omega}(\boldsymbol{u}). \tag{1.10}$$

Δw_i^{BS} is defined by (2.4.11) of Batchelor (1967, p. 87) as the volume integral (1.11) taken over the whole of space, including the region $\mathscr{V}_B$ in the interior the body. Here, ξ denotes the distance between the position vector x_l and the integration variable x_l' namely $\xi^2 = (x_l - x_l')(x_l - x_l')$. The disturbance vorticy $\Delta\omega_j$ is analytically continued into the interior of the body by solving the Laplace problem for a potential function ψ, where $\psi_{,j} = \Delta\omega_j$, which satisfies the normal boundary condition $\psi_{,j}n_j|_B = \Delta\omega_j n_j|_B$ on the surface of the body.

$$\Delta w_i^{BS}(\boldsymbol{u}) = \frac{1}{4\pi}\varepsilon_{ijk}\int \Delta\omega_j'(\boldsymbol{u})\frac{\partial}{\partial x_k'}(\xi^{-1})\,\mathrm{d}\mathscr{V}'. \tag{1.11}$$

Here, the functional notation $(\boldsymbol{u})$ is being used to make explicit the relationship of the quantities with the velocity field u_i.

We can now explore the interrelationship between the rotational velocity and drift. As explained in Batchelor (1967, pp. 274, 275), the vortex tubes are frozen into an inviscid incompressible flow and the local vorticity field $\omega_j(\boldsymbol{u})$, therefore, is generated from the ambient vorticity Ω_i by the distortion of the vortex tubes caused by the velocity field u_i. The relationship is defined by (5.3.9) of Batchelor as the tensor product of the distortion tensor $\partial x_i(\boldsymbol{u})/\partial x_j^{-\infty}$ and the ambient vorticity Ω_j, namely

$$\omega_i(\boldsymbol{u}) = \frac{\partial x_i(\boldsymbol{u})}{\partial x_j^{-\infty}}\Omega_j. \tag{1.12}$$

Here, the particle positions $x_i(\boldsymbol{u})$ at time t are viewed as a function of their starting coordinates $x_i^{-\infty}$. The disturbance vorticity is, therefore, related to the drift vector by substituting (1.5) and (1.8) into (1.12) to give

$$\Delta\omega_i(\boldsymbol{u}) = \frac{-\partial d_i(\boldsymbol{u})}{\partial x_j^{-\infty}}\Omega_j = -\Omega\frac{\partial d_i(\boldsymbol{u})}{\partial x_3^{-\infty}}. \tag{1.13}$$

To make the analysis tractable, Lighthill linearizes his analysis with respect to w_i by assuming that the strain rate is weak or more precisely that the strain-induced velocity $a_B\Omega$ is much smaller than the relative velocity U, namely

$$a_B\Omega/U \ll 1, \quad O\left(a_B^2\Omega^2/U^2\right) \equiv 0. \tag{1.14}$$

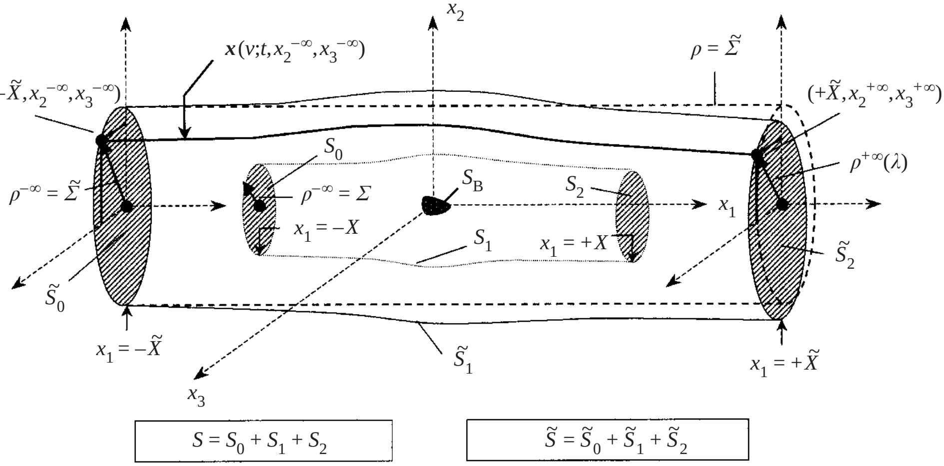

FIGURE 1. Coordinate systems, the body $\mathscr{S}_B$ and the surfaces $\mathscr{S}$ and $\tilde{\mathscr{S}}$.

Here, a_B is a length scale associated with the body which can be defined in terms of the volume $\mathscr{V}_B$ of the body as $a_B = \mathscr{V}_B^{1/3}$. Under this assumption, therefore, all terms whose order is proportional to Ω^2 are neglected. Thus, because the disturbance vorticity $\Delta\omega_i(\boldsymbol{u})$ and the rotational velocity field w_i are of order $O(\Omega)$ and $O(a_B\Omega)$, respectively, then $d_i(\boldsymbol{u}) = d_i(\boldsymbol{v}) + O(a_B^2\Omega/U)$ and we can adopt the following approximation $\Delta\omega_i(\boldsymbol{v})$ for the disturbance vorticity $\Delta\omega_i(\boldsymbol{u})$ with a negligible error of $O(a_B\Omega^2/U)$

$$\Delta\omega_i(\boldsymbol{v}) = -\Omega\,\partial d_i(\boldsymbol{v})/\partial x_3^{-\infty}. \tag{1.15}$$

Here, the functional dependence of d_i upon v_i indicates that we need only take account of the distortion caused by the irrotational velocity field v_i when calculating the rotational disturbance velocity. It is now evident from (1.15) why the analysis of the rotational velocity field is integrally related to the study of drift in the corresponding irrotational flow. Furthermore, the vorticity $\Delta\omega_i^{+\infty}$ in the trailing vortex far downstream of the body, as discussed in Lighthill (1956, p. 35), is asymptotically independent of x_1 and is related to the total drift D_i by

$$\Delta\omega_i^{+\infty} = -\Omega\,\partial D_i/\partial x_3^{-\infty} + O(a_B\Omega^2/U). \tag{1.16}$$

The final part of our argument is to determine the force on the body using the analytical approach employed by Auton (1987). In his § 6, Auton applies the divergence theorem to the momentum equation, written in the form $(1/\rho_0)p_{,i} + (u_iu_j)_{,j} = 0$ for incompressible flows, in the large volume $\tilde{\mathscr{V}} - \mathscr{V}_B$ surrounding the body $\mathscr{V}_B$. Here $\tilde{\mathscr{V}}$, as shown in figure 1, is defined as being enclosed by the far upstream plane $x_1^{-\infty} = -\tilde{X}$, the far downstream plane $x_1^{+\infty} = +\tilde{X}$ and the stream surface of the irrotational velocity field v_i originating from the circle $\rho^{-\infty} = \tilde{\Sigma}$. His analysis requires that the streamwise length of $\tilde{\mathscr{V}}$ be much greater than its radius and, therefore, $a_B \ll \tilde{\Sigma} \ll \tilde{X}$. We then obtain (1.17) which is equivalent to (6.1) of Auton (1987) for the force f_i on the body in terms of the limit, as $\tilde{\Sigma}$ and $\tilde{X} \to +\infty$, of an integral over the 'asymptotic' surface $\tilde{\mathscr{S}}$ of $\tilde{\mathscr{V}}$

$$\frac{1}{\rho_0}f_i = \lim_{\tilde{X},\tilde{\Sigma}\to+\infty} \int_{\tilde{\mathscr{S}}} \left(-\frac{1}{\rho_0}pn_i - u_iu_jn_j\right) \mathrm{d}s. \tag{1.17}$$

Note, it is assumed that the ratio $\tilde{\Sigma}/\tilde{X} \to 0$ as $\tilde{\Sigma}$ and $\tilde{X} \to +\infty$. Now from Batchelor (1967, p. 405) the limit of integral (1.17) vanishes identically in the absence of vorticity ($\Omega = 0$) since then $u_i = v_i$ and the force f_i becomes equal to that on a body in an irrotational flow with uniform steady ambient velocity. It is then possible to simplify the integrand of (1.17), under the assumption that terms whose orders are proportional to Ω^2 can be neglected, to involve only the asymptotic values of the rotational disturbance velocity. In the case of a sphere, Auton arrives at his equation (6.16) which is equivalent to (1.18) below

$$\frac{1}{\rho_0} f_i = -\mathscr{V}_B\, C_M\, U \Omega \delta_{2i}. \tag{1.18}$$

Note that the sign in Auton's (6.16) is misprinted.

The analysis in our paper will parallel the three steps discussed above. First, the application of Darwin's work to an arbitrarily shaped body. Secondly, the derivation of the asymptotic form for the rotational disturbance velocity field for an arbitrarily shaped body as was derived for a sphere by Lighthill (1956, 1957). Finally, the generalization of Auton's argument (1987, § 6) to determine the expression for the lift force.

2. Problem formulation

We shall aim to employ tensor notation whenever possible. In doing so we have found it very helpful to introduce the notation $T_{i\neq 1}$ to represent that part of the tensor T_i for which $i \neq 1$. Thus, we can write $T_i = T_1\delta_{1i} + T_{i\neq 1}$. This notation allows us to express tensors in terms of their components parallel $T_1\delta_{1i}$ and perpendicular $T_{i\neq 1}$ to the direction of motion. In particular, therefore, the position vector x_i has the unique decomposition

$$x_i = x_1\delta_{1i} + x_{i\neq 1} = x_1\delta_{1i} + \rho\lambda_i. \tag{2.1}$$

Here, λ_i is the cylindrical polar unit angular vector defined as

$$\lambda_i = (0, \cos\lambda,\ \sin\lambda). \tag{2.2}$$

We will find that the angular vector λ_j will occur in many of the integrals over the interval $0 < \lambda < 2\pi$ when the following identities apply

$$\int_0^{2\pi} \lambda_i \,\mathrm{d}\lambda = 0; \quad \int_0^{2\pi} \lambda_i\lambda_j \,\mathrm{d}\lambda = \pi\delta_{ij\neq 1}; \quad \int_0^{2\pi} \lambda_i\lambda_j\lambda_k \,\mathrm{d}\lambda = 0. \tag{2.3}$$

To avoid repetitive use of integral signs we shall conduct much of our analysis in terms only of the integrands. Thus, we have adopted the equivalence notation ($\equiv$) between integrands to denote identity of the corresponding integrals. This amounts to dropping terms that are a function of the azimuthal angle λ whose integrals are identically zero.

Now we consider various notational aspects of the disturbance velocity fields. The irrotational disturbance velocity Δv_i will be defined as the gradient of the disturbance velocity potential $\Delta\varphi$ thus

$$\Delta v_i = U\Delta\varphi_{,i}. \tag{2.4a}$$

Note that in order to ensure $\Delta\varphi$ is single-valued, and the corresponding Laplace problem correctly posed, the shape of the body must be such that the surrounding fluid region is singly connected. See, for example, Batchelor (1967, § 2.7). From p. 121 of Batchelor it follows that the disturbance potential and the irrotational disturbance velocity have the following asymptotic approximations at large radial distances from the body

$$\Delta\varphi \sim -c_l x_l r^{-3}, \quad \Delta v_i \sim -c_l U(\delta_{li} r^{-3} - 3x_l x_i r^{-5}). \tag{2.4b}$$

This particular definition of the velocity potential has been chosen to be consistent with that used by Lighthill (1956) so as to ensure that the asymptotic coefficients c_l have the dimensions of the body's volume, namely $c_l = O(a_B^3)$. As explained in § 1, the rotational velocity w_i given by (1.10) can be approximated to order $O(a_B^2 \Omega^2/U)$ by

$$w_i = -\Omega x_2 \delta_{1i} + \Delta w_i^{BS}(\boldsymbol{v}) + \Delta v_i^{\Omega}(\boldsymbol{v}). \tag{2.5a}$$

Note that the irrotational velocity $\Delta v_i^{\Omega}(\boldsymbol{v})$ could have a non-zero volume flux at the surface of the body, even though the body is rigid. This is because the zero-flux velocity boundary condition $w_i n_i|_B = 0$ does not exclude the possibility that the boundary volume flux induced by the Biot-Savart integral $\Delta w_i^{BS}(\boldsymbol{v})$ is non zero. Writing c^{Ω} as the volume flux then, as explained in Batchelor (1967, p. 121), the leading-order asymptotic form for $\Delta v_i^{\Omega}(\boldsymbol{v})$ is given by

$$\Delta v_i^{\Omega} \sim \Omega c^{\Omega}(x_i r^{-3}). \tag{2.5b}$$

We shall parallel the argument of Lighthill and express $\Delta w_i^{BS}(\boldsymbol{v})$ as the sum of three contributions $\Delta w_{i(\mathrm{I})}$, $\Delta w_{i(\mathrm{II})}$, $\Delta w_{i(\mathrm{III})}$ corresponding to three subdivisions $\mathscr{V}_{(\mathrm{I})}$, $\mathscr{V}_{(\mathrm{II})}$ and $\mathscr{V}_{(\mathrm{III})}$ of the integration domain (the whole of space) of the Biot-Savart integral. Thus, we write

$$\Delta w_i^{BS}(\boldsymbol{v}) = \Delta w_{i\,(\mathrm{I})} + \Delta w_{i\,(\mathrm{II})} + \Delta w_{i\,(\mathrm{III})}. \tag{2.6}$$

Since the streamlines $x_i(\boldsymbol{v})$ of the irrotational velocity field v_i span the whole of space outside of the body then the regions can be defined in terms of the streamlines $x_i(\boldsymbol{v})$ as follows. First region $\mathscr{V}_{(\mathrm{I})}$ corresponds to streamlines that remain at a large polar radius from the body which is defined in terms of their starting coordinates as

$$\mathscr{V}_{(\mathrm{I})} = \{x_i(\boldsymbol{v})| -\infty < x_1 +\infty; \rho^{-\infty} \geqslant \Sigma\}. \tag{2.7}$$

Here, Σ is a large radius relative to that of the body, but, as will become apparent later in the argument, Σ must be much smaller than the equivalent quantity $\tilde{\Sigma}$ that defines the radius of the asymptotic surface to be used in determining the force. Thus,

$$a_B \ll \Sigma \ll \tilde{\Sigma}. \tag{2.8}$$

The second region $\mathscr{V}_{(\mathrm{II})}$ corresponds to the volume enclosed by the body together with the remaining streamlines that originate far upstream, but stop at a far-downstream but finite position ($x_1 = +X$) in the trailing vortex where the vorticity field has become independent of x_1. Thus, $\mathscr{V}_{(\mathrm{II})}$ is defined by

$$\mathscr{V}_{(\mathrm{II})} = \{x_i(\boldsymbol{v})|-\infty < x_1 < +X; \rho^{-\infty} < \Sigma\} \cup \mathscr{V}_B. \tag{2.9}$$

Here, it is a necessary requirement of our analysis that X is very much greater than Σ but also that both Σ and X are very much smaller than the equivalent quantities $\tilde{\Sigma}$ and $\tilde{X}$ defining the asymptotic volume $\tilde{\mathscr{V}}$ and its surface $\tilde{\mathscr{S}}$. Thus,

$$a_B \ll \Sigma \ll X \ll \tilde{\Sigma} \ll \tilde{X}. \tag{2.10}$$

Finally, region $\mathscr{V}_{(\mathrm{III})}$ is defined by the remainder of the whole of space which importantly includes that part of the trailing vortex where the vorticity is independent of x_1

$$\mathscr{V}_{(\mathrm{III})} = \{x_i(\boldsymbol{v})| +X < x_1 < +\infty; \rho^{-\infty} < \Sigma\}. \tag{2.11}$$

We can now define the asymptotic volume $\tilde{\mathscr{V}}$, as shown in figure 1. In view of the particular shape of the asymptotic volume chosen, it should be noted that the following argument is only strictly valid for a body whose cross-stream section has an aspect ratio of order one. The cross-stream to axial aspect ratio can necessarily be much larger because $\tilde{\Sigma} \ll \tilde{X}$. Thus, a long slender body is consistent with the analysis provided that the body's principal axis is aligned with the undisturbed flow. The principles of the mathematical argument can be applied to other body shapes provided the shape of the asymptotic volume is consistent with that of the body and care is taken when evaluating the conditionally convergent integrals in the limit as the surface of $\tilde{\mathscr{V}}$ is allowed to tend to infinity. For the purposes of our analysis, therefore, the hydraulic radius $a_B(=\mathscr{V}_B^{1/3})$ will be used as the characteristic length scale of the body in recognition of the implicit constraints on its shape, as described above.

In terms of the streamlines $x_i(\boldsymbol{v})$ of the irrotational velocity field v_i, then $\tilde{\mathscr{V}}$ is defined as

$$\tilde{\mathscr{V}} = \{x_i(\boldsymbol{v}) | -\tilde{X} < x_1 < +\tilde{X}; \rho^{-\infty} < \tilde{\Sigma}\}. \tag{2.12}$$

The asymptotic surface $\tilde{\mathscr{S}}$ of the volume $\tilde{\mathscr{V}}$ is then defined as the sum of three parts

$$\tilde{\mathscr{S}} = \tilde{\mathscr{S}}_0 + \tilde{\mathscr{S}}_1 + \tilde{\mathscr{S}}_2. \tag{2.13}$$

Here, $\tilde{\mathscr{S}}_1$ is the stream surface originating from the far upstream circle $\rho^{-\infty} = \tilde{\Sigma}$ and defined by

$$\tilde{\mathscr{S}}_1 = \{x_i(\boldsymbol{v}) | -\tilde{X} < x_1 < +\tilde{X}; \rho^{-\infty} = \tilde{\Sigma}\}. \tag{2.14}$$

$\tilde{\mathscr{S}}_0$ and $\tilde{\mathscr{S}}_2$ are the far upstream and downstream disks defined by

$$\tilde{\mathscr{S}}_0 = \{x_i(\boldsymbol{v}) | x_1 = -\tilde{X}; \rho^{-\infty} < \tilde{\Sigma}\}, \quad \tilde{\mathscr{S}}_2 = \{x_i(\boldsymbol{v}) | x_1 = +\tilde{X}; \rho^{-\infty} < \tilde{\Sigma}\}. \tag{2.15}$$

It is important to note that the normal vector n_i to the stream surface $\tilde{\mathscr{S}}_1$ comprises two components. First, the normal λ_i to the circular cylinder $\rho = \tilde{\Sigma}$ corresponding to the ambient uniform velocity $U\delta_{1i}$ and, secondly, a component Δn_i corresponding to the irrotational disturbance velocity Δv_i. It is argued in Appendix A that, on $\tilde{\mathscr{S}}_1$ the position vector can be approximated to second order by

$$x_i(\boldsymbol{v}) \sim \tilde{x}_i - \tilde{d}_i, \tag{2.16a}$$

where $\tilde{x}_i$ are the first-order asymptotic streamlines given by

$$\tilde{x}_i = x_1\delta_{1i} + \tilde{\Sigma}\lambda_i. \tag{2.16b}$$

Here, $\tilde{d}_i$ is the following approximate form for the drift vector which corresponds to equation (16) of Lighthill

$$\tilde{d}_i(\boldsymbol{v}) = \int_{-\infty}^{x_1} -\Delta\varphi_{,i}|_{\tilde{x}}\, \mathrm{d}x_1 = O\left(a_B^3 \tilde{\Sigma}^{-2}\right). \tag{2.17}$$

It is also argued in Appendix A that

$$n_i = \lambda_i + \Delta n_i|_{\tilde{x}} \quad \text{where} \quad \Delta n_i|_{\tilde{x}} = O\left(a_B^3 \tilde{\Sigma}^{-3}\right). \tag{2.18}$$

Here subscript $\tilde{\boldsymbol{x}}$ denotes evaluation of functions on the asymptotic streamline $\tilde{x}_i$.

In the course of our argument, we shall derive expressions involving surface integrals over both the upstream $\tilde{\mathscr{S}}_0$ and downstream disks $\tilde{\mathscr{S}}_2$. The differential surface elements on the upstream $\mathrm{d}s^{-\infty}$ and downstream disks $\mathrm{d}s^{+\infty}$, however, correspond to the far upstream and downstream ends of a stream tube of the velocity field v_i. Since the volume flux in the x_1-direction is conserved, the fluid being incompressible, then we have the relationship

$$v_1^{-\infty} \mathrm{d}\mathscr{s}^{-\infty} = v_1^{+\infty} \mathrm{d}\mathscr{s}^{+\infty}. \tag{2.19}$$

By the definition of v_i given by (1.3), however, $v_1^{-\infty} = v_1^{+\infty} = U$, from which it follows that the differential surface elements are equal and thus

$$\mathrm{d}\mathscr{s}^{-\infty} = \mathrm{d}\mathscr{s}^{+\infty}. \tag{2.20}$$

This relationship is crucial to the argument since it allows integrations over the downstream disk $\tilde{\mathscr{S}}_2$ to be transformed into integrals over the upstream disk $\tilde{\mathscr{S}}_0$.

3. Darwin's theorem for an arbitrarily shaped body

Since Darwin's drift-volume occurs in our final expression, (6.21), for the lift force, we shall derive in this section an alternative expression in terms of the added mass coefficient tensor. Following the argument of Darwin (1953, § 8), we first obtain an identity for his drift-volume (the left hand side of the identity below) by substituting for the definition of the total drift given by (1.6) to obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D_i \,\mathrm{d}x_2^{-\infty}\,\mathrm{d}x_3^{-\infty} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} - U\Delta\varphi_{,i}\,\mathrm{d}t\,\mathrm{d}x_2^{-\infty}\,\mathrm{d}x_3^{-\infty}. \tag{3.1}$$

Here, the total drift D_i is considered to be a function of the far upstream off-axis coordinates and, therefore, $D_i = D_i(x_2^{-\infty}, x_3^{-\infty})$. Now note that every fluid particle that moves from $x_i(t) \to x_i(t) + \Delta v_i\,\mathrm{d}t$ in the time interval $t \to t + \mathrm{d}t$ originated from $x_1^{-\infty}$ where it would have moved from $x_1^{-\infty} \to x_1^{-\infty} + \mathrm{d}x_1^{-\infty}$ in the corresponding starting time interval $t^{-\infty} \to t^{-\infty} + \mathrm{d}t$. We can, therefore, replace $U\mathrm{d}t$ in the right-hand side of (3.1) by $\mathrm{d}x_1^{-\infty}$ to obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D_i \,\mathrm{d}x_2^{-\infty}\,\mathrm{d}x_3^{-\infty} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} - \Delta\varphi_{,i}\,\mathrm{d}x_1^{-\infty}\,\mathrm{d}x_2^{-\infty}\,\mathrm{d}x_3^{-\infty}. \tag{3.2}$$

Here, $\Delta\varphi_{,i}$ is still evaluated at time t at the point x_i(t). Furthermore, since the fluid is incompressible, the fluid volume element $\mathrm{d}x_1^{-\infty}\,\mathrm{d}x_2^{-\infty}\,\mathrm{d}x_3^{-\infty}$ which originates at the starting time $t^{-\infty}$ is deformed into the fluid volume element $\mathrm{d}\upsilon$ during the time interval $t^{-\infty} \to t$. Necessarily, the summed volume elements $\mathrm{d}\upsilon$ comprise the whole of the volume surrounding the body $\mathscr{V}_B$ which we denote $\mathscr{V}_\infty - \mathscr{V}_B$. Substituting the identity $\mathrm{d}x_1^{-\infty}\mathrm{d}x_2^{-\infty}\mathrm{d}x_3^{-\infty} = \mathrm{d}\upsilon$ into (3.2), we arrive at the equivalent of Darwin's equation (8.8), namely

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} D_i \,\mathrm{d}x_2^{-\infty}\,\mathrm{d}x_3^{-\infty} = \int_{\mathscr{V}_\infty - \mathscr{V}_B} - \Delta\varphi_{,i}\,\mathrm{d}\upsilon. \tag{3.3}$$

We now wish to evaluate the right-hand side of (3.3), but we must first carefully consider how this should be done in view of the conditional convergence, as discussed by Darwin, of the multiple integral. As explained in § 1, the need to evaluate the left-hand side of (3.3) arises naturally in both the derivation of the asymptotic approximation of the rotational disturbance velocity and also in the evaluation of the force integral (1.17) on the far downstream disk $\tilde{\mathscr{S}}_2$ of the asymptotic surface $\tilde{\mathscr{S}}$. Thus, the particular evaluation of Darwin's theorem for our analysis must parallel that of the main argument. For our purpose we can, therefore, apply Darwin's theorem rigorously in the form of the identity

$$\lim_{\tilde{X},\tilde{\Sigma}\to+\infty} \int_{\tilde{\mathscr{S}}_0} D_i\,\mathrm{d}\mathscr{s} = \lim_{\tilde{X},\tilde{\Sigma}\to+\infty} \int_{\tilde{\mathscr{V}}-\mathscr{V}_B} - \Delta\varphi_{,i}\,\mathrm{d}\upsilon. \tag{3.4}$$

Here, it is necessary to ensure that the ratio $\tilde{\Sigma}/\tilde{X} \to 0$ whilst taking the limit. This ensures that the limit is much more advanced in the streamwise direction than in the off-axis direction which is consistent with evaluating the innermost integral of the right-hand side of (3.1) first, before evaluating the double integral for the drift-volume.

In our argument, we shall employ both the identity (6.4.28) of Batchelor (1967, p. 407) for the acceleration reaction in terms of the disturbance velocity potential $\Delta\varphi$ and identity (6.4.29) (Batchelor, p. 408) for the fluid impulse $\mathscr{I}_i$ in terms of the added mass coefficient tensor C_{ij} (denoted α_{ij} by Batchelor). When these two identities are combined the fluid impulse becomes

$$\mathscr{I}_i = \int_{\mathscr{S}_B} U\Delta\varphi n_i \,\mathrm{d}s = \mathscr{V}_B U C_{i1}. \tag{3.5}$$

Note that only the added mass coefficient tensor terms C_{i1} appear in the identity because, in our formulation, the ambient velocity is equal to $U\delta_{1j}$. Applying the divergence theorem to the right-hand side of (3.4) and substituting (3.5) we arrive at

$$\int_{\tilde{\mathscr{V}}-\mathscr{V}_B} -\Delta\varphi_{,i}\,\mathrm{d}v = \mathscr{V}_B C_{i1} - \int_{\tilde{\mathscr{S}}} \Delta\varphi n_i \,\mathrm{d}s. \tag{3.6}$$

We now substitute into (3.6) the asymptotic approximations (3.7) for $\Delta\varphi$ and n_i on $\tilde{\mathscr{S}}_1$ and the far upstream and downstream disks $\tilde{\mathscr{S}}_0$ and $\tilde{\mathscr{S}}_2$. The approximations on $\tilde{\mathscr{S}}_1$ are obtained by taking Taylor expansions in the off-axis direction about the streamlines $\tilde{x}_i$ of the uniform flow and substituting the bounds given by (2.17) and (2.18) to give

$$\Delta\varphi|_{\tilde{\mathscr{S}}_1} \sim -c_k(x_k r^{-3})|_{\tilde{x}} - \Delta\varphi_{,k}|_{\tilde{x}} d_k = -c_k(x_k r^{-3})|_{\tilde{x}} + O\left(a_B^6\tilde{\Sigma}^{-2}r^{-3}\right), \tag{3.7a}$$

$$n_i|_{\tilde{\mathscr{S}}_1} = \lambda_i + O\left(a_B^3\tilde{\Sigma}^{-3}\right), \quad \Delta\varphi|_{\tilde{\mathscr{S}}_0} = \Delta\varphi|_{\tilde{\mathscr{S}}_2} = O\left(a_B^3\tilde{X}^{-2}\right). \tag{3.7b}$$

Noting that

$$\int_{-\infty}^{+\infty} (r^{-3})|_{\tilde{x}}\,\mathrm{d}x_1 = \int_{-\infty}^{+\infty} \left[x_1^2 + \tilde{\Sigma}^2\right]^{-3/2}\mathrm{d}x_1 = 2\tilde{\Sigma}^{-2}$$

and also that on $\tilde{\mathscr{S}}_1$ the differential surface element is given by $\mathrm{d}s = \tilde{\Sigma}\,\mathrm{d}x_1\,\mathrm{d}\lambda$ we arrive at

$$\int_{\tilde{\mathscr{V}}-\mathscr{V}_B} -\Delta\varphi_{,i}\,\mathrm{d}v = \mathscr{V}_B C_{i1} + \tilde{\Sigma}\int_{\tilde{\mathscr{S}}_1} c_k(x_k r^{-3})|_{\tilde{x}}\lambda_i\,\mathrm{d}x_1\,\mathrm{d}\lambda + O\left(a_B^6\tilde{\Sigma}^{-3}\right) + O\left(a_B^3\tilde{\Sigma}^2/\tilde{X}^2\right). \tag{3.8}$$

Now, substitute $(x_k)|_{\tilde{x}} = x_1\delta_{1k} + \tilde{\Sigma}\,\lambda_k$ and note that for integration with respect to λ over the interval $0 < \lambda < 2\pi$ then odd functions of λ can be dropped so that $\lambda_i r^{-3} \equiv 0$ and $\lambda_i\lambda_k r^{-3} \equiv \delta_{ik\neq 1} r^{-3}/2$. It follows that as $\tilde{\Sigma}$ and $\tilde{X} \to +\infty$,

$$\int_{\tilde{\mathscr{S}}_1} c_k(x_k r^{-3})|_{\tilde{x}}\lambda_i\,\mathrm{d}s \to \tilde{\Sigma}^2\int_{-\infty}^{+\infty}\int_0^{2\pi} c_{k\neq 1}(r^{-3})|_{\tilde{x}}\lambda_k\lambda_i\,\mathrm{d}\lambda\,\mathrm{d}x_1 = 2\pi c_{i\neq 1}. \tag{3.9}$$

Combining (3.8) and (3.9) with the relationship between c_k and the added mass coefficient tensor C_{ij} given by (6.4.18) of Batchelor (1967, p. 403), namely $c_{i\neq 1} = -\mathscr{V}_B/(4\pi)C_{i\neq 11}$ (note the difference in sign because in Batchelor's notation the body is moving and the fluid stationary) we find

$$\int_{\tilde{\mathscr{V}}-\mathscr{V}_B} -\Delta\varphi_{,i}\,\mathrm{d}v \to \mathscr{V}_B C_{i1} + 2\pi c_{i\neq 1} = \mathscr{V}_B C_{i1} - \mathscr{V}_B(0, C_{21}, C_{31})/2. \tag{3.10}$$

Thus, we finally obtain the required identity for the drift-volume

$$\lim_{\tilde{X},\tilde{\Sigma}\to+\infty}\int_{\tilde{\mathscr{S}}_0} D_i \, \mathrm{d}\mathscr{s} = \lim_{\tilde{X},\tilde{\Sigma}\to+\infty}\int_{\tilde{\mathscr{V}}-\mathscr{V}_B} -\Delta\varphi_{,i}\,\mathrm{d}v = \mathscr{V}_B\left(C_{11},\tfrac{1}{2}C_{21},\tfrac{1}{2}C_{31}\right). \tag{3.11}$$

Identity (3.11) is seen to agree with equation (3.1) of Darwin and equation (14) of Lighthill (1956) in the case of a sphere when $\mathscr{V}_B = 4\pi/3a_B^3$, $C_{11} = C_M = \frac{1}{2}$ and $C_{21} = C_{31} = 0$.

4. Asymptotic approximation of the rotational disturbance velocity Δw_i^{BS} on $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$

4.1. *The asymptotic split of* Δw_i^{BS} *into* $\Delta w_i^{①}$ *and* $\Delta w_i^{②}$

As explained in §2, we wish to approximate Δw_i^{BS} essentially by splitting the infinite domain of integration of the Biot-Savart integral of $\Delta\omega_i$ into the three subdomains $\mathscr{V}_{(\mathrm{I})}$, $\mathscr{V}_{(\mathrm{II})}$ and $\mathscr{V}_{(\mathrm{III})}$. The conditional convergence of the Biot-Savart integral, however, prevents us from proceeding directly on this basis. We therefore adopt the procedure described in Lighthill (1956, p. 37) and separate out from $\Delta\omega_i$ its asymptotic value $\Delta\tilde{\omega}_i$ on streamlines that remain far from the body, given by equation (18) of Lighthill as

$$\Delta\tilde{\omega}_i(\boldsymbol{v}) = \frac{-\Omega\,\partial\,\tilde{d}_i(\boldsymbol{v})}{\partial x_3^{-\infty}} = \frac{\Omega\,\partial}{\partial x_3^{-\infty}}\left[\int_{-\infty}^{x_1}\Delta\varphi_{,i}|_{\tilde{\boldsymbol{x}}}\,\mathrm{d}x_1\right] = \Omega\int_{-\infty}^{x_1}\Delta\varphi_{,i3}|_{\tilde{\boldsymbol{x}}}\,\mathrm{d}x_1. \tag{4.1}$$

Note that Lighthill points out in a footnote to his p. 37 that if the argument is not progressed rigorously in this way then a different and incorrect result is obtained. Lighthill's procedure, as developed rigorously later in the section, ensures that the difference between $\Delta\omega_i$ and its asymptotic value $\Delta\tilde{\omega}_i$ decays very rapidly as $\rho\to+\infty$, namely like $\Delta\omega_i - \Delta\tilde{\omega}_i = O(\Omega a_B^5\rho^{-5})$. Thus, the radial contribution of this error to the Biot-Savart integrals for $\Delta w_{i(\mathrm{II})}$ and $\Delta w_{i(\mathrm{III})}$ is, roughly speaking, of order $O(\Omega a_B^5\Sigma^{-4})$ and, therefore, negligible in the limit as $\tilde{\Sigma}\to+\infty$. If the argument is not progressed in this way, there are finite contributions in the limit as $\tilde{\Sigma}$ and $\tilde{X}\to+\infty$ that arise from the interface between $\mathscr{V}_{(\mathrm{I})}$ and $\mathscr{V}_{(\mathrm{II})}+\mathscr{V}_{(\mathrm{III})}$ whose rigorous treatment would substantially complicate the analysis and whose non-rigorous treatment, as indicated by Lighthill, would lead to an incorrect result. For the same reason, it is important to split the domain of the Biot-Savart integral in such a way that it is consistent with our limiting procedure, namely $a_B/\Sigma\to 0$, $\Sigma/X\to 0$, $X/\tilde{\Sigma}\to 0$, $\tilde{\Sigma}/\tilde{X}\to 0$ as $\tilde{\Sigma}$ and $\tilde{X}\to+\infty$. This in the reason why we have defined the surfaces $\tilde{\mathscr{S}}$ and $\mathscr{S}$ to be nested, as shown in figure 1, and also the volumes $\mathscr{V}_{(\mathrm{I})}$, $\mathscr{V}_{(\mathrm{II})}$ and $\mathscr{V}_{(\mathrm{III})}$ as having interfaces that are coincident $\mathscr{S}_1$ and $\mathscr{S}_2$. To rigorously define the separation of $\Delta\tilde{\omega}_i$ from $\Delta\omega_i$ we shall follow Lighthill by defining $\Delta\omega_i$ to be split into two components $\Delta\omega_i^{①}$ and $\Delta\omega_i^{②}$ where

$$\Delta\omega_i = \Delta\omega_i^{①} + \Delta\omega_i^{②}. \tag{4.2}$$

The function $\Delta\omega_i^{①}$ is then defined as an analytical continuation of $\Delta\tilde{\omega}_i$ from large cylindrical polar radius, $\rho^{-\infty} > \rho_B$ say, into the whole of space but in such a way that $\Delta\omega_i^{①}$ is identically zero in an inner cylinder, $\rho^{-\infty} < \rho_A$ say, which is still at a large distance from the body. The position of the inner and outer cylinder are arbitrary, but the argument is much simplified if we choose them to be within an infinitesimally small distance from the stream-cylinder $\rho^{-\infty} = \Sigma$, which we have chosen as the interface between $\mathscr{V}_{(\mathrm{I})}$ and $\mathscr{V}_{(\mathrm{II})}+\mathscr{V}_{(\mathrm{III})}$. Thus, we can now proceed rigorously provided that we evaluate $\Delta\omega_i$ in $\mathscr{V}_{(\mathrm{I})}$ by considering the evaluation of $\Delta\omega_i^{①}$ and $\Delta\omega_i^{②}$ in $\mathscr{V}_{(\mathrm{I})}$ separately and employ $\Delta\omega_i^{②}$ instead of $\Delta\omega_i$ in $\mathscr{V}_{(\mathrm{II})}+\mathscr{V}_{(\mathrm{III})}$ (because $\Delta\omega_i^{①}=0$). The key difference is that $\Delta\omega_i^{②}$ decays very rapidly as $\rho\rightarrow+\infty$ in $\mathscr{V}_{(\mathrm{I})}$. The rapidity of this decay can be estimated by considering the next highest order approximation to $\Delta\omega_i(\boldsymbol{v})$ on the distant streamlines $\tilde{\boldsymbol{x}}_j$ employing the bound $\tilde{d}_l\sim O(a_B^3\rho^{-2})$, derived in Appendix A, to obtain

$$\Delta\varphi_{,i}\sim\Delta\varphi_{,i}|_{\tilde{x}}+\Delta\varphi_{,il\neq1}|_{\tilde{x}}\,(x_{l\neq1}-\tilde{x}_{l\neq1})\sim\Delta\varphi_{,i}|_{\tilde{x}}-\Delta\varphi_{,il\neq1}|_{\tilde{x}}\;\tilde{d}_{l\neq1}$$
$$\sim\Delta\varphi_{,i}|_{\tilde{x}}+O\left(a_B^6\rho^{-2}r^{-4}\right).\quad(4.3)$$

When (4.3) is substituted into identity (4.1) for the asymptotic form of the disturbance vorticity $\Delta\omega_i(\boldsymbol{v})$ and noting that

$$\int_{x_1}^{+\infty}(r^{-4})|_{\tilde{x}}\,\mathrm{d}x_1<\int_{-\infty}^{+\infty}(r^{-4})|_{\tilde{x}}\,\mathrm{d}x_1=\int_{-\infty}^{+\infty}\left[x_1^2+\rho^2\right]^{-2}\mathrm{d}x_1=O(\rho^{-3})$$

as $\rho\rightarrow+\infty$, we find

$$\Delta\omega_i^{②}\sim\Delta\omega_i-\Delta\tilde{\omega}_i=O\left(\Omega a_B^5\rho^{-5}\right).\quad(4.4)$$

Having now defined the split in the disturbance vorticity $\Delta\omega_i$ into $\Delta\omega_i^{①}$ and $\Delta\omega_i^{②}$, the corresponding split in Δw_i^{BS} is defined by the Biot-Savart integrals of $\Delta\omega_i^{①}$ and $\Delta\omega_i^{②}$ respectively, as

$$\Delta w_i^{BS}=\Delta w_i^{①}+\Delta w_i^{②}.\quad(4.5)$$

4.2. *The asymptotic approximation of $\Delta w_{i\,(\mathrm{I})}$ as $\rho\rightarrow+\infty$ and $|x_1|\rightarrow+\infty$*

First, we consider the asymptotic approximation of $\Delta w_{i(\mathrm{I})}$ as $\rho\rightarrow+\infty$. As explained in §4.1, to evaluate $\Delta w_{i\,(\mathrm{I})}$ we must evaluate $\Delta w^{①}_{i\,(\mathrm{I})}$ and $\Delta w^{②}_{i\,(\mathrm{I})}$ separately. By definition, $\Delta\omega_i^{①}$ is equal to $\Delta\tilde{\omega}_i$ in $\mathscr{V}_{(\mathrm{I})}$, where $\Delta\tilde{\omega}_i$ as defined in (4.1), denotes $\Delta\omega_i$ evaluated on the streamlines that remain far from the body. By the definition of the Biot-Savart integral, the curl of $\Delta w_i^{①}$ is identically equal to $\Delta\tilde{\omega}_i$ in the region $\mathscr{V}_{(\mathrm{I})}$. Also by applying Lighthill's argument for his equations (16)–(19) we find that the curl of $\Delta\tilde{w}_i$ is equal to $\Delta\tilde{\omega}_i$ where

$$\Delta\tilde{w}_i=\Omega\,\varepsilon_{i3k}\tilde{d}_k=-\Omega(\tilde{d}_2,-\tilde{d}_1,0).\quad(4.6)$$

Equation (4.6) is seen to be identically equal to (19) of Lighthill (1956) by noting that $\tilde{d}_1=-\Delta\varphi$ and, in his notation, $\Omega=-A$. It follows, therefore, that $\Delta\tilde{w}_i$ is equal to the highest-order term in the asymptotic approximation of $\Delta w^{①}_{i\,(\mathrm{I})}$ as $\rho\rightarrow+\infty$. It remains, therefore, to approximate $\Delta w^{②}_{i\,(\mathrm{I})}$.

Consider the asymptotic behaviour of $\Delta\omega_i(\boldsymbol{v})$ in the far downstream limit as $x_1\rightarrow+\infty$. First, note that the streamlines $x_i(\boldsymbol{v})$ tend towards the straight lines

$$x_i(\boldsymbol{v})\rightarrow\tilde{\boldsymbol{x}}_i^+=x_1\,\delta_{1i}+x_{i\neq1}^{+\infty}.\quad(4.7)$$

Denoting evaluation on the far downstream streamlines $\tilde{\boldsymbol{x}}^+$ by $|_{\tilde{x}^+}$ we can write

$$\Delta\omega_i^{②}|_{\tilde{x}^+}=\Delta\omega_i|_{\tilde{x}^+}-\Delta\tilde{\omega}_i|_{\tilde{x}^+}=\left(\Delta\omega_i|_{\tilde{x}^+}-\Delta\omega_i^{+\infty}|_{\tilde{x}^+}\right)+\left(\Delta\omega_i^{+\infty}|_{\tilde{x}^+}-\Delta\tilde{\omega}_i|_{\tilde{x}^+}\right).\quad(4.8a)$$

It now follows, by employing the argument of Appendix C to both terms on the right-hand side of (4.8a), that as $x_1\rightarrow+\infty$

$$\Delta\omega_i^{②}|_{\tilde{x}^+}=\Delta\omega_i|_{\tilde{x}^+}-\Delta\tilde{\omega}_i|_{\tilde{x}^+}=O\left(\Omega a_B^3|x_1|^{-3}\right).\quad(4.8b)$$

A similar argument can be used to show that on the far upstream streamlines $\tilde{\boldsymbol{x}}$ then, as $x_1 \to -\infty$

$$\Delta\omega_i^{②}|_{\tilde{x}^+} = \Delta\omega_i = O\big(\Omega a_B^3 |x_1|^{-3}\big). \tag{4.8c}$$

Note that both bounds in (4.8b) and (4.8c) are uniform in ρ. Now consider the behaviour of $\Delta\omega_i(\boldsymbol{v})$ as $\rho \to +\infty$. Combining the three bounds (4.4), (4.8b) and (4.8c) for the behaviours of $\Delta\omega_i$ both as $|x_1| \to +\infty$ and $\rho \to +\infty$ we obtain

$$\Delta\omega_i^{②} = \Delta\omega_i - \Delta\tilde{\omega}_i = O\big(\Omega a_B^3 |x_1|^{-3}\big), \tag{4.9a}$$

and

$$\Delta\omega_i^{②} = \Delta\omega_i - \Delta\tilde{\omega}_i = O\big(\Omega a_B^5 \rho^{-5}\big). \tag{4.9b}$$

Note that the bound in (4.9b) is uniform in x_1. We can now substitute the bounds (4.9) for $\Delta\omega_i^{②}$ into the Biot-Savart integral for $\Delta w_{i\,(\mathrm{I})}^{②}$. To do this, we split the x_1 integration range into two parts. In the first region $|x_1| > X$ then $\Delta\omega_i^{②}$ is of order $O(\Omega a_B^3 |x_1|^{-3})$ and in the second region $|x_1| < X$ then $\Delta\omega_i^{②}$ is of order $O(\Omega a_B^5 \rho^{-5})$. Substituting these two bounds into the Biot-Savart integral for $\Delta w_{i\,(\mathrm{I})}^{②}$ and integrating the bounds with respect to $\rho\, \mathrm{d}\rho$ over the interval $\Sigma < \rho < +\infty$ we find

$$\Delta w_{i\,(\mathrm{I})}^{②} = O(a_B \Omega \Sigma^2 / X^2) + O\big(\Omega a_B^3 \Sigma^{-3} X\big). \tag{4.10}$$

Note that both of the bounds in (4.10) tend to zero as Σ and $X \to +\infty$ since we are free to choose the ratio $a_B/\Sigma = (\Sigma/X)^\beta$ provided $\beta > 0$. It follows that provided $\beta > \frac{1}{2}$ then $\Sigma^{-3}X = a_B^{-2}(a_B/\Sigma)^2 X/\Sigma = a_B^{-2}(\Sigma/X)^{2\beta-1}$ and the term of order $O(\Omega a_B^3 \Sigma^{-3} X) = O(a_B \Omega \Sigma^{2\beta-1}/X^{2\beta-1}) \to 0$ as Σ and $X \to +\infty$. Thus, if we choose $\frac{1}{2} < \beta < \frac{3}{2}$ we recover the result given by (19) of Lighthill (1956) as $\rho \to +\infty$, namely

$$\Delta w_{i\,(\mathrm{I})} = \Delta w_{i\,(\mathrm{I})}^{①} + \Delta w_{i\,(\mathrm{I})}^{②} = \Delta\tilde{w}_i + O(a_B \Omega \Sigma^{2\beta-1}/X^{2\beta-1}) \sim \Omega \varepsilon_{i3k}\, \tilde{d}_k. \tag{4.11a}$$

We can now approach the asymptotic approximation of $\Delta w_{i\,(\mathrm{I})}$ as $|x_1| \to +\infty$ in an entirely analogous way to that above to show also that as $|x_1| \to +\infty$

$$\Delta w_{i\,(\mathrm{I})} = \Delta w_{i\,(\mathrm{I})}^{①} + \Delta w_{i\,(\mathrm{I})}^{②} = \Delta\tilde{w}_i + O(a_B \Omega \Sigma^{2\beta-1}/X^{2\beta-1}) \sim \Omega\, \varepsilon_{i3k}\, \tilde{d}_k. \tag{4.11b}$$

4.3. *The asymptotic approximation of* $\Delta w_{i(\mathrm{II})}$ *as* $r \to +\infty$

The following analysis applies in the asymptotic limit as $r \to +\infty$ and, therefore, the results are separately valid in the two limiting cases studied later, namely $\rho \to +\infty$ and $|x_1| \to +\infty$. The first step in approximating the Biot-Savart integral for $\Delta w_{i\,(\mathrm{II})}$, or equivalently $\Delta w_{i\,(\mathrm{II})}^{②}$, is to truncate the integral over the infinite volume $\mathscr{V}_{(\mathrm{II})}$ to one over a finite volume $\mathscr{V}$. Here, as shown in figure 1, $\mathscr{V}$ and $\mathscr{S}(=\mathscr{S}_0 + \mathscr{S}_1 + \mathscr{S}_2)$ are defined identically to $\tilde{\mathscr{V}}$ and $\tilde{\mathscr{S}}$ in equations (2.12)–(2.15) except that Σ and X are used instead of $\tilde{\Sigma}$ and $\tilde{X}$. Employing the bound (4.9) for $\Delta\omega_i^{②}$ as $|x_1| \to +\infty$ we can, therefore, approximate the Biot-Savart integral for $\Delta w_{i(\mathrm{II})}^{②}$ by (4.12) where here we have purposely chosen to write the partial derivative of ξ with respect to x_k and not the integration variable x'_k, where $\xi^2 = (x_l - x'_l)(x_l - x'_l)$

$$\Delta w_{i(\mathrm{II})} = \frac{-1}{4\pi} \varepsilon_{ijk} \left[\int_{\mathscr{V}} \Delta\omega'^{②}_j \frac{\partial}{\partial x_k}(\xi^{-1}) \mathrm{d}v' \right] + O(a_B \Omega \Sigma^2 / X^2). \tag{4.12}$$

Now since the volume $\mathscr{V}$ is bounded by $\rho^{-\infty} < \Sigma$ and $|x_1| < X$, then for large radius r we can approximate $\xi \sim r$ and take the partial derivative outside the integral to obtain the equivalent identity to that of (20) in Lighthill (1956), once corrected in sign, namely

$$\Delta w_{i(\mathrm{II})} \sim -\frac{1}{4\pi}\varepsilon_{ijk}\left[\int_{\mathscr{V}} \Delta\omega'^{②}_{j}\,\mathrm{d}v'\right](r^{-1})_{,k}. \tag{4.13}$$

Note the error in the sign of (20) of Lighthill (1956) is explained in Lighthill (1957). We now follow the argument used by Lighthill (1956) to obtain his equation (21). Substitute the identity $(x'_j\Delta\omega^{②}_l)_{,l} = \Delta\omega'^{②}_j$ into (4.13) and apply the divergence theorem to obtain

$$\Delta w_{i(\mathrm{II})} \sim \frac{-1}{4\pi}\varepsilon_{ijk}\left[\int_{\mathscr{S}_0+\mathscr{S}_1+\mathscr{S}_2} x'_j\Delta\omega'^{②}_l n'_l\,\mathrm{d}s'\right](r^{-1})_{,k}. \tag{4.14}$$

Note on $\mathscr{S}_0$ that $x'_j|_{\mathscr{S}_0} = -X\delta_{1j}$, $n'_l|_{\mathscr{S}_0} = -\delta_{1l}$ and $\Delta\omega^{②}_1|_{\mathscr{S}_0} \sim O(\Omega a_B^3|x'_1|^{-3})$; on $\mathscr{S}_1$ that $x'_j|_{\mathscr{S}_1} \sim x'_1\delta_{1j} + \Sigma\lambda'_j$, $n'_l|_{\mathscr{S}_1} \sim \lambda'_l$ and $\Delta\omega^{②}_l|_{\mathscr{S}_1} = O(\Omega a_B^5\Sigma^{-5})$; on $\mathscr{S}_2$ that $x'_j|_{\mathscr{S}_2} \sim X\delta_{1j} + x'^{+\infty}_{j\neq 1}$, $n'_l|_{\mathscr{S}_2} = \delta_{1l}$ and $\Delta\omega'^{②}_1|_{\mathscr{S}_2} \sim -\Omega\,\partial d'_1/\partial x'^{-\infty}_3 + O(\Omega a_B^3|x'_1|^{-3})$. The only finite contribution to (4.14), therefore, comes from the far downstream disk $\mathscr{S}_2$. The disk $\mathscr{S}_0$ contributes an error of $O(a_B^3\Omega\Sigma^2/X^2)$ to the inner integral of (4.14) which becomes negligible as Σ and $X \to +\infty$. When $x'_j\Delta\omega'^{②}_l$ is integrated over $\mathscr{S}_1$ it yields an error $O(a_B{}^5\Omega\Sigma^{-4}X^2)$ to the inner integral of (4.14). Using the same reasoning as used to derive (4.11), we are free to choose the ratio $a_B/\Sigma = (\Sigma/X)^\beta$ whereby, provided $\beta > 1$, then $\Sigma^{-4}X^2 = a_B^{-2}(a_B/\Sigma)^2(X/\Sigma)^2 = a_B^{-2}(\Sigma/X)^{2\beta-2}$ and the term of order $O(a_B^5\Omega\,\Sigma^{-4}X^2) = O(a_B{}^3\Omega\,\Sigma^{2\beta-2}/X^{2\beta-2}) \to 0$ as Σ and $X \to +\infty$. Thus, as Σ and $X \to +\infty$

$$\int_{\mathscr{S}_0+\mathscr{S}_1+\mathscr{S}_2} x'_j\Delta\omega'^{②}_l\, n'_l\,\mathrm{d}s' \sim -\Omega\int_{\mathscr{S}_2}\left(\frac{\partial\, d'_1}{\partial x'^{-\infty}_3}x'_j\right)\bigg|_{\mathscr{S}_2}\mathrm{d}s'^{+\infty}. \tag{4.15}$$

Now changing the integration variables from $\mathrm{d}s'^{+\infty}$ to $\mathrm{d}s'^{-\infty}$ using (2.20), writing $\mathrm{d}s'^{-\infty} = \mathrm{d}x'^{-\infty}_2\,\mathrm{d}x'^{-\infty}_3$ and $x'_j|_{\mathscr{S}_2} \sim X\delta_{1j} + x'^{+\infty}_{j\neq 1}$ we obtain

$$\int_{\mathscr{S}_0+\mathscr{S}_1+\mathscr{S}_2} x'_j\,\Delta\omega'^{②}_l\, n'_l\,\mathrm{d}s' \sim -\Omega\delta_{1j}X\int_{\mathscr{S}_0}\frac{\partial}{\partial x'^{-\infty}_3}(d'_1|_{\mathscr{S}_2})\,\mathrm{d}x'^{-\infty}_2\,\mathrm{d}x'^{-\infty}_3$$
$$-\,\Omega\int_{\mathscr{S}_0}\left(\frac{\partial d'_1}{\partial x'^{-\infty}_3}x'_{j\neq 1}\right)\bigg|_{\mathscr{S}_2}\mathrm{d}s'^{-\infty}. \tag{4.16}$$

The first integral on the right-hand side of (4.16) can be integrated once with respect to $x'^{-\infty}_3$. Using the identity $d'_1|_{\mathscr{C}_2} = \tilde{d}'_1|_{\mathscr{C}_2} = -\Delta\varphi'|_{\mathscr{C}_2} = O(a_B^3X^{-2})$, where here we have denoted evaluation on the perimeter boundary contour of $\mathscr{S}_2$ by $|_{\mathscr{C}_2}$, then $X\int_{\mathscr{S}_0}\partial/\partial x'^{-\infty}_3(d'_1|_{\mathscr{S}_2})\,\mathrm{d}x'^{-\infty}_2\,\mathrm{d}x'^{-\infty}_3 = O(a_B^3\Sigma/X)$. Finally, letting Σ and $X \to +\infty$ then the drift $d'_1|_{\mathscr{S}_2}$ evaluated on $\mathscr{S}_2$, tends towards the total drift D'_1 and, therefore, $(\partial d'_1/\partial x'^{-\infty}_3 x'_{j\neq 1})|_{\mathscr{S}_2} \to \partial D'_1/\partial x'^{-\infty}_3 x'^{+\infty}_{j\neq 1}$ and we obtain the limiting identity

$$\int_{\mathscr{S}_0+\mathscr{S}_1+\mathscr{S}_2} x'_j\Delta\omega^{②}_l\, n'_l\,\mathrm{d}s' \to -\Omega\int_{\mathscr{S}_0}\frac{\partial D'_1}{\partial x'^{-\infty}_3}x'^{+\infty}_{j\neq 1}\,\mathrm{d}s''^{-\infty}. \tag{4.17}$$

Substituting (4.17) into (4.14) for $\Delta w_{i\,(\mathrm{II})}$ we obtain

$$\Delta w_{i\,(\mathrm{II})} \sim \frac{1}{4\pi}\Omega\varepsilon_{ijk}\left[\int_{\mathscr{S}_0}\frac{\partial D'_1}{\partial x'^{-\infty}_3}x'^{+\infty}_{j\neq 1}\,\mathrm{d}s'^{-\infty}\right](r^{-1})_{,k}. \tag{4.18}$$

Since $(r^{-1})_{,k} = -r^{-3}x_k = -x_1 r^{-3}\delta_{1k} - \rho r^{-3}\lambda_k$ (4.18) can alternatively be written

$$\Delta w_{i\,(\mathrm{II})} \sim \frac{-1}{4\pi}\Omega(\varepsilon_{ij1}x_1 r^{-3} + \delta_{1i}\varepsilon_{1jk}\rho r^{-3}\lambda_k)\int_{\mathscr{S}_0} \frac{\partial D_1'}{\partial x_3'^{-\infty}} x_{j\neq 1}'^{+\infty}\,\mathrm{d}s'^{-\infty}. \tag{4.19}$$

The result (4.18) is our generalization of the corrected equation (22) of Lighthill (1956). The identity between our (4.18) and the corrected (22) of Lighthill in his case of a sphere will be proved in the discussion of § 7.

4.4. *The asymptotic approximation of* $\Delta w_{i\,(\mathrm{III})}$ *as* $\rho \to +\infty$ *and* $|x_1| \to +\infty$

For the limiting case $\rho \to +\infty$, we shall approximate the Biot-Savart integral for $\Delta w_{i(\mathrm{III})}$, or equivalently $\Delta w^{②}_{i\,(\mathrm{III})}$ since $\Delta\omega_i^{①} = 0$ in $\mathscr{V}_{(\mathrm{III})}$. Noting that by the definition of $\mathscr{V}_{(\mathrm{III})}$ then $x_1' > +X$, the disturbance vorticity $\Delta\omega_i'^{②}$ is asymptotically independent of x_1' and only a function of the far-downstream off-axis coordinates $x_{j\neq 1}'^{+\infty}$. We can, employing the bound derived in Appendix C, substitute $\Delta\omega_i'^{②} = \Delta\omega_j'^{+\infty} + O(\Omega a_B^3 x_1'^{-3})$ and take the term $\Delta\omega_j'^{+\infty}$ outside the integral with respect to x_1' to obtain

$$\Delta w_{i\,(\mathrm{III})} = \frac{-1}{4\pi}\varepsilon_{ijk}\int_{\mathscr{S}_2}\Delta\omega_j'^{+\infty}\int_X^{+\infty}\frac{\partial}{\partial x_k}(\xi^{-1})\,\mathrm{d}x_1'\,\mathrm{d}s'^{+\infty} + O(a_B\Omega\Sigma^2/X^2). \tag{4.20}$$

Note that we have changed the sign of the integral by replacing the partial derivative with respect to the integration variable $\partial/\partial x_k'$ by the partial derivative with respect to the independent variable $\partial/\partial x_k$. Since our interest in the value of $\Delta w_{i\,(\mathrm{III})}$ lies at a large radial distance ρ ($\gg X \gg \Sigma$) we can substitute the asymptotic approximation $\boldsymbol{B}_k$ $(=\boldsymbol{B}_1\delta_{1k} + \boldsymbol{B}_{k\neq 1})$ for $\int_X^{+\infty}(\partial/\partial x_k)(\xi^{-1})\,\mathrm{d}x_1'$ derived in Appendix B, whilst at the same time change the integration variables $\mathrm{d}s'^{+\infty}$ on disk $\mathscr{S}_2$ to $\mathrm{d}s'^{-\infty}$ on disk $\mathscr{S}_0$. Substituting $\Delta\omega_j'^{+\infty}$ for its explicit expression in terms of the total drift given by (1.16), then $\Delta\omega_j'^{+\infty} = -\Omega\,\partial D_j'/\partial x_3'^{-\infty}$ and we obtain

$$\Delta w_{i\,(\mathrm{III})} \sim \frac{1}{4\pi}\Omega\int_{\mathscr{S}_0}\left\{\varepsilon_{ij1}\frac{\partial D_{j\neq 1}'}{\partial x_3'^{-\infty}}\boldsymbol{B}_1 + \varepsilon_{ijk\neq 1}\frac{\partial D_j'}{\partial x_3'^{-\infty}}\boldsymbol{B}_{k\neq 1}\right\}\mathrm{d}s'^{-\infty}. \tag{4.21}$$

Before substituting the expression for $\boldsymbol{B}_k$ in Appendix B, note that any terms in $\boldsymbol{B}_k$ that are independent of the integration variable $x_{j\neq 1}'^{-\infty}$ can be neglected. This is because these terms can be integrated with respect to $x_3'^{-\infty}$ and since $D_j' = O(a_B^3\Sigma^{-2})$ on the boundary contour $\mathscr{C}_2$ of disk $\mathscr{S}_2$ then they will only make an order $O(a_B^2\Omega\Sigma^{-1})$ contribution to $\Delta w_{i\,(\mathrm{III})}$. Now substituting (B 4) for $\boldsymbol{B}_1$ and (B 5*c*) for $\boldsymbol{B}_{k\neq 1}$ we obtain

$$\begin{aligned}\Delta w_{i\,(\mathrm{III})} \sim{}& \frac{1}{4\pi}\Omega\varepsilon_{ij1}(\rho r^{-3}\lambda_l)\int_{\mathscr{S}_0}\frac{\partial D_{j\neq 1}'}{\partial x_3'^{-\infty}}x_{l\neq 1}'^{+\infty}\,\mathrm{d}s'^{-\infty}\\ &+\frac{1}{4\pi}\Omega\varepsilon_{ijk\neq 1}(\rho^{-2}[1 + x_1 r^{-1}]\{\delta_{kl\neq 1} - 2\lambda_k\lambda_l\} - x_1 r^{-3}\lambda_k\lambda_l)\int_{\mathscr{S}_0}\frac{\partial D_j'}{\partial x_3'^{-\infty}}x_{l\neq 1}'^{+\infty}\,\mathrm{d}s'^{-\infty}\end{aligned} \tag{4.22}$$

The identity (4.22) for $\Delta w_{i\,(\mathrm{III})}$ is not directly comparable with Lighthill, but can be shown to agree with his equation for $\Delta w_{i(\mathrm{II})} + \Delta w_{i(\mathrm{III})}$ given by (85) in Lighthill (1957). The proof of the equality between Lighthill's (85) for a spherical body and that derived here in (4.22) will be addressed in the discussion of § 7.

For the limiting case $|x_1| \to +\infty$, then $\rho' \ll \rho \ll |x_1|$ and we note from the identities (B 1) and (B 2*b*) of Appendix B the following uniform bounds in ρ, where $\eta^2 = (x_{l\neq 1} - x_{l\neq 1}')(x_{l\neq 1} - x_{l\neq 1}')$.

$$\boldsymbol{B}_{k=1} = O(|x_1|^{-1}), \tag{4.23a}$$

$$\boldsymbol{B}_{k\neq 1} = 2(x'_{k\neq 1} - x_{k\neq 1})\eta^{-2} + O(\rho|x_1|^{-2}) = -2\frac{\partial}{\partial x_k}(\log\eta) + O(\rho|x_1|^2). \qquad (4.23b)$$

Thus, substituting the identity $\Delta\omega_i^{+\infty} = -\Omega\,\partial D_i/\partial x_3^{-\infty}$ from (1.16) and the bounds (4.23) into (4.20) for $\Delta w_{i\,(\mathrm{III})}$ we find as $|x_1| \to +\infty$ that

$$\Delta w_{i(\mathrm{III})} = \frac{-1}{2\pi}\Omega\varepsilon_{ijk\neq 1}\int_{\mathscr{S}_2}\frac{\partial D'_j}{\partial x_3'^{-\infty}}\frac{\partial}{\partial x_k}(\log\eta)|_{\tilde{\mathscr{S}}_2}\,\mathrm{d}s'^{-\infty} + O(\Omega\Sigma^3/X^2) + O(a_B\Omega\Sigma^2/X^2). \qquad (4.24a)$$

Finally, substituting $a_B/\Sigma = (\Sigma/X)^\beta$, we find that provided we choose $0 < \beta < 2$ (which is possible in addition to satisfying the constraints $\frac{3}{2} > \beta > \frac{1}{2}$ and $\beta > 1$ in §§4.2 and 4.3, respectively) then as Σ and $X \to +\infty$ we obtain the two-dimensional Biot-Savart integral, namely,

$$\Delta w_{i(\mathrm{III})} = \frac{-1}{2\pi}\Omega\varepsilon_{ijk\neq 1}\int_{\mathscr{S}_2}\frac{\partial D'_j}{\partial x_3'^{-\infty}}\frac{\partial}{\partial x_k}(\log\eta)|_{\tilde{\mathscr{S}}_2}\,\mathrm{d}s'^{+\infty} + O(a_B\Omega\Sigma^{2-\beta}/X^{2-\beta}). \qquad (4.24b)$$

5. The lift force expressed as an integral of Δw_i over $\tilde{\mathscr{S}}_1$ and the disks $\tilde{\mathscr{S}}_0$ and $\tilde{\mathscr{S}}_2$

5.1. *The lift force expressed as an integral of w_i over $\tilde{\mathscr{S}}$*

As explained in the §1, we shall proceed in the same way as §6 of Auton (1987) and employ his identity (6.1) for the force on the body expressed as an integral over $\tilde{\mathscr{S}}$. Note that in the proof of his (6.1) the surface $\tilde{\mathscr{S}}$ can be chosen to have any shape, provided it encloses the body. The particular shape of the asymptotic surface $\tilde{\mathscr{S}}$ used in our argument is defined in §2 as comprising the far upstream disk $\tilde{\mathscr{S}}_0$, the far downstream disk $\tilde{\mathscr{S}}_2$ and the stream surface $\tilde{\mathscr{S}}_1$ of the velocity field v_i. Thus, we can write Auton's equation (6.1) in the following form, (5.1), where we have introduced the notation $|_{\boldsymbol{x}(\boldsymbol{v})}$ to make explicit the evaluation of functions on the streamlines of v_i and not u_i.

$$\frac{1}{\rho_0}f_i = \lim_{\tilde{X},\tilde{\Sigma}\to+\infty}\int_{\tilde{\mathscr{S}}}\left\{\frac{-1}{\rho_0}p|_{\boldsymbol{x}(\boldsymbol{v})}n_i - (u_iu_j)|_{\boldsymbol{x}(\boldsymbol{v})}n_j\right\}\mathrm{d}s. \qquad (5.1)$$

For brevity, we shall omit the integral signs in the following argument and use the equivalence notation ($\equiv$) to denote equality under the integral. Starting with the right-hand term in the integrand of (5.1), we express u_i as the sum of its irrotational v_i and rotational w_i components as defined in (1.9). Noting that $w_i = O(a_B\Omega)$, we obtain

$$(u_iu_j)|_{\boldsymbol{x}(\boldsymbol{v})} \equiv (v_iv_j)|_{\boldsymbol{x}(\boldsymbol{v})} + (v_iw_j + v_jw_i)|_{\boldsymbol{x}(\boldsymbol{v})} + O\left(a_B^2\Omega^2\right). \qquad (5.2)$$

To approximate $p|_{\boldsymbol{x}(\boldsymbol{v})}$ we employ the Bernoulli identity on the streamlines of u_i that originate from the far upstream starting positions $x_i^{-\infty}$ to obtain

$$\frac{-1}{\rho_0}p|_{\boldsymbol{x}(\boldsymbol{u})} = \tfrac{1}{2}\left(u_ju_j - u_j^{-\infty}u_j^{-\infty}\right)\big|_{\boldsymbol{x}(\boldsymbol{u})} = \tfrac{1}{2}\left(v_jv_j - v_j^{-\infty}v_j^{-\infty}\right)\big|_{\boldsymbol{x}(\boldsymbol{u})} + \left(v_jw_j - v_j^{-\infty}w_j^{-\infty}\right)|_{\boldsymbol{x}(\boldsymbol{u})} + O\left({a_B}^2\Omega^2\right). \qquad (5.3)$$

Here, the pressure $p^{-\infty}$ at the far upstream position has been taken to be identically zero since its inclusion does not change the value of (5.1). To approximate $p|_{\boldsymbol{x}(\boldsymbol{v})}$, we must relate the pressure at x_1 on the streamline $x_i(\boldsymbol{u})$ to that at x_1 on the neighbouring streamline $x_i(\boldsymbol{v})$. To do this, we take a Taylor expansion about the streamline $x_i(\boldsymbol{v})$

in the x_2- and x_3-directions from which it follows that the two terms in (5.3) can be approximated, respectively, by

$$\left(v_j w_j - v_j^{-\infty} w_j^{-\infty}\right)\big|_{x(u)} = \left(v_j w_j - v_j^{-\infty} w_j^{-\infty}\right)\big|_{x(v)} + O\left(a_B{}^2\Omega^2\right), \tag{5.4a}$$

$$\tfrac{1}{2}\left(v_j v_j - v_j^{-\infty} v_j^{-\infty}\right)\big|_{x(u)} = \tfrac{1}{2}\left(v_j v_j - v_j^{-\infty} v_j^{-\infty}\right)\big|_{x(v)} + (v_j v_{j,k\neq 1})|_{x(v)}[x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v})]. \tag{5.4b}$$

The streamline displacement $x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v})$ is normal to the direction of uniform flow and is given to order $O(a_B^3\Omega^2/U^2)$ by the integral of the rotational disturbance velocity components $\Delta w_{k\neq 1}$ along the streamlines $x_i(\boldsymbol{v})$. Importantly, therefore, $x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v}) = O(a_B^2\Omega/U)$. Note that the streamline displacement is normal to the ambient velocity $U_k = (U - \Omega x_2)\delta_{1k}$. Using a similar argument to that used in Appendix A to derive the bound $d_k(\boldsymbol{v}) = O(a_B^3\tilde{\Sigma}^{-2})$ for the drift vector d_k corresponding to the streamline displacement caused by Δv_k, it is possible to argue that as $\tilde{\Sigma} \to +\infty$

$$x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v}) = O\left(a_B{}^3\Omega\tilde{\Sigma}^{-1}/U\right). \tag{5.5}$$

Combining the bound (5.5) with $(v_j v_{j,k\neq 1})|_{x(v)} = O(U^2 a_B^3 r^{-4})$ we obtain from (5.4*b*) that

$$\tfrac{1}{2}\left(v_j v_j - v_j^{-\infty} v_j^{-\infty}\right)\big|_{x(u)} = \tfrac{1}{2}\left(v_j v_j - v_j^{-\infty} v_j^{-\infty}\right)\big|_{x(v)} + O\left(a_B{}^6 U\Omega\tilde{\Sigma}^{-1} r^{-4}\right). \tag{5.6}$$

The error term $O(a_B^6 U\Omega\tilde{\Sigma}^{-1}r^{-4})$ in (5.6), when integrated over the upstream and downstream disks $\tilde{\mathscr{S}}_0$ and $\tilde{\mathscr{S}}_2$, contributes $O(a_B^6 U\Omega\tilde{\Sigma}/\tilde{X}^4)$ to $1/\rho_0 f_i$ and, when integrated over the stream surface $\tilde{\mathscr{S}}_1$, contributes $O(a_B^6 U\Omega\tilde{\Sigma}^{-3})$. Combining (5.3), (5.4) and (5.6), we find that, in the limit as $\tilde{\Sigma}$ and $\tilde{X} \to +\infty$, $p|_{x(u)}$ can be approximated over the whole of $\tilde{\mathscr{S}}$ by

$$\frac{-1}{\rho_0} p|_{x(u)} \equiv \tfrac{1}{2}\left(v_j v_j - v_j^{-\infty} v_j^{-\infty}\right)\big|_{x(v)} + \left(v_j w_j - v_j^{-\infty} w_j^{-\infty}\right)\big|_{x(v)}. \tag{5.7}$$

It remains to approximate the pressure $p|_{x(v)}$ on the streamlines $x_i(\boldsymbol{v})$. Again we take the Taylor expansion about $x_i(\boldsymbol{v})$ to obtain the leading – order approximation $-1/\rho_0 p|_{x(u)} = -1/\rho_0 p|_{x(v)} - 1/\rho_0 p_{,k\neq 1}|_{x(v)}[x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v})]$. Substituting for $-1/\rho_0 p_{,k}$ from the momentum equation $-1/\rho_0 p_{,k} = u_l u_{k,l}$ and substituting the bound (5.5) for the streamline displacement we find that

$$\begin{aligned}\frac{-1}{\rho_0} p|_{x(u)} &= \frac{-1}{\rho_0} p|_{x(v)} + (u_l u_{k,l})|_{x(v)}[x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v})] \\ &= -\frac{1}{\rho_0} p|_{x(v)} + (v_l\, v_{k,l})|_{x(v)}[x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v})] + O\left(a_B^2\Omega^2\right). \end{aligned} \tag{5.8}$$

Finally, substituting the bounds $(v_l v_{k,l})|_{x(v)} = O(a_B^3 U^2 r^{-4})$ and again $x_{k\neq 1}(\boldsymbol{u}) - x_{k\neq 1}(\boldsymbol{v}) = O(a_B^3\Omega\tilde{\Sigma}^{-1}/U)$ while letting $\tilde{\Sigma}$ and $X \to +\infty$ we determine the following approximation over the whole of $\tilde{\mathscr{S}}$:

$$\frac{-1}{\rho_0} p|_{x(u)} \equiv -\frac{1}{\rho_0} p|_{x(v)} + O\left(a_B{}^2\Omega^2\right). \tag{5.9}$$

We can now combine the approximations (5.2), (5.7) and (5.9) to give the following equivalence for the integrand $-(1/\rho_0)p|_{x(v)}n_i - (u_i u_j)|_{x(v)}n_j$ of (5.1) over the whole of $\tilde{\mathscr{S}}$:

$$\begin{aligned}\frac{-1}{\rho_0} p|_{x(v)}n_i - (u_i u_j)|_{x(v)}n_j \equiv{}& \left\{\tfrac{1}{2}\left(v_j v_j - v_j^{-\infty} v_j^{-\infty}\right)\big|_{x(v)}n_i - (v_i v_j)|_{x(v)}n_j\right\} \\ &+ \left\{\left(v_j w_j - v_j^{-\infty} w_j^{-\infty}\right)\big|_{x(v)}n_i - (v_i w_j + v_j w_i)|_{x(v)}n_j\right\} + O\left(a_B^2\Omega^2\right). \end{aligned} \tag{5.10}$$

In the absence of any vorticity in the flow, the second curly bracketed term on the right-hand side of (5.10) would vanish and the remaining expression would then give the integrand of the force integral (5.1) for the case of a body in the irrotational velocity field v_i. This case then corresponds to that discussed in Batchelor (1967, p. 405) in which he argues that the force on the body is identically zero. It follows that the integral of the first curly bracketed term on the right-hand side of (5.10) must equal zero, whereupon we arrive at the equivalence

$$\frac{-1}{\rho_0} p|_{x(v)} n_i - (u_i u_j)|_{x(v)} n_j \equiv \left\{ \left(v_j w_j - v_j^{-\infty} w_j^{-\infty} \right) \big|_{x(v)} n_i - (v_i w_j + v_j w_i)|_{x(v)} n_j \right\} + O\left(a_B^2 \Omega^2\right). \quad (5.11)$$

Finally, substituting (5.11) into the force integral (5.1), we obtain the identity for the force on the body to order $O(a_B^4 \Omega^2)$

$$\frac{1}{\rho_0} f_i = \lim_{\tilde{X}, \tilde{\Sigma} \to +\infty} \int_{\tilde{\mathscr{S}}} \left\{ \left(v_j w_j - v_j^{-\infty} w_j^{-\infty} \right) n_i - (v_i w_j + v_j w_i) n_j \right\} \mathrm{d}s. \quad (5.12)$$

Here, the integrand is implicitly taken to be evaluated on the streamlines $x_i(\boldsymbol{v})$. Auton (1987) does not provide an explicit identity with which to compare this result, but the reader will be able to identify a number of the steps used in the argument above in Auton's discussion at the beginning of his § 6.

5.2. *The contribution to the force integral from* Δw_i *on the stream surface* $\tilde{\mathscr{S}}_1$

For the sake of brevity we shall omit the integrals over $\tilde{\mathscr{S}}_1$ in the following and use the equivalence notation ($\equiv$) to denote equality under the integral. We shall also drop the $|_{x(v)}$ subscript since it is now implicit that we are evaluating functions on the stream surface $\tilde{\mathscr{S}}_1$ and in particular, therefore, $v_j n_j|_{\tilde{\mathscr{S}}_1} = 0$ to give

$$\left\{ \frac{-1}{\rho_0} p n_i - u_i u_j n_j \right\} \equiv \left(v_j w_j - v_j^{-\infty} w_j^{-\infty} \right) n_i - (v_i w_j) n_j. \quad (5.13)$$

Following the approach in § 6 of Auton (1987), we first write the terms in (5.13) as perturbations about their values at the far upstream starting positions thus

$$v_i w_j = \left(v_i - v_i^{-\infty}\right)\left(w_j - w_j^{-\infty}\right) + v_i^{-\infty}\left(w_j - w_j^{-\infty}\right) + \left(v_i - v_i^{-\infty}\right) w_j^{-\infty} + v_i^{-\infty} w_j^{-\infty}. \quad (5.14)$$

Note the following relationships which follow from (1.9), (2.16) and (2.17)

$$v_i^{-\infty} = U \delta_{1i}, \quad w_j^{-\infty} = -\Omega x_2^{-\infty} \delta_{1j}, \quad v_i - v_i^{-\infty} = \Delta v_i, \quad (5.15a)$$

$$w_j - w_j^{-\infty} = -\Omega \left(x_2 - x_2^{-\infty}\right) \delta_{1j} + \Delta w_j = \Omega \tilde{d}_2 \delta_{1j} + \Delta w_j. \quad (5.15b)$$

Substituting (5.15) into (5.14) we find

$$\left(v_j w_j - v_j^{-\infty} w_j^{-\infty}\right) = \Omega \tilde{d}_2 \Delta v_1 + \Delta v_j \Delta w_j + U \Omega \tilde{d}_2 + U \Delta w_1 - \Omega x_2^{-\infty} \Delta v_1, \quad (5.16a)$$

$$(v_i w_j) n_j = \left(-U \Omega x_2^{-\infty} \delta_{1i} + U \Omega \tilde{d}_2 \delta_{1i} - \Omega x_2^{-\infty} \Delta v_i + \Omega \tilde{d}_2 \Delta v_i\right) n_1 + U \delta_{1i} \Delta w_j n_j + \Delta v_i \Delta w_j n_j. \quad (5.16b)$$

At this point, we must remind the reader, as shown in Appendix A, that $n_j = \lambda_j + \Delta n_j$, and therefore, $n_1 = \Delta n_1$. The reader should note that Auton, in his § 6 for the case of a spherical body, makes no reference to the contribution from the perturbation Δn_j in the normal vector. Its omission, however, makes no difference to Auton's final result. Its inclusion for an arbitrarily shaped body, on the other hand, is essential because it results in terms that have a non-zero contribution to the force integral. Moreover, because of the bounds $\Delta v_j = O(Ua_B^3 r^{-3})$, $\Delta w_j = O(a_B^3 \Omega \tilde{\Sigma}^{-2})$, $\tilde{d}_i = O(a_B^3 \tilde{\Sigma}^{-2})$ and $\Delta n_i = O(a_B^3 \tilde{\Sigma}^{-3})$, we can drop the products $\Delta v_j \Delta w_j$, $\tilde{d}_2 \Delta v_i$, $\Delta w_1 \Delta n_i$, $\Delta w_j \Delta n_j$, $\tilde{d}_2 \Delta n_i$, $\tilde{d}_2 \Delta v_j$, $x_2^{-\infty} \Delta v_1 \Delta n_i$ and $x_2^{-\infty} \Delta v_i \Delta n_1$ that arise in (5.16) because they make a negligible contribution to the integral over $\tilde{\mathscr{S}}_1$. Note that the term $\tilde{d}_2 \Delta n_i$ is of order $O(a_B^6 \tilde{\Sigma}^{-5})$ and $\Delta w_1 \Delta n_i$ and $\Delta w_j \Delta n_j$ are of order $O(a_B^6 \Omega \tilde{\Sigma}^{-5})$ and, therefore, contribute $O(a_B^6 U \Omega \tilde{\Sigma}^{-4} \tilde{X})$ to the integral which can be shown to be negligible in the limit as $\tilde{\Sigma}$ and $\tilde{X} \to +\infty$ using a similar argument to that used for (4.10)–(4.11). Again noting that $n_j = \lambda_j + \Delta n_j$, and $n_1 = \Delta n_1$, we find

$$\left\{ \frac{-1}{\rho_0} p n_i - u_i u_j n_j \right\} \bigg|_{\tilde{\mathscr{S}}_1} \equiv U(\Delta w_1 \lambda_i - \Delta w_j \lambda_j \delta_{1i}) + \left(U\Omega \tilde{d}_2 - \Omega x_2^{-\infty} \Delta v_1 \right) \lambda_i - \left(U\Omega \tilde{d}_2 \delta_{1i} - \Omega x_2^{-\infty} \Delta v_i - U\Omega x_2^{-\infty} \delta_{1i} \right) \Delta n_1. \quad (5.17)$$

We shall now obtain an alternative expression for Δn_1 by analysing the asymptotic approximation to the exact identity $v_j n_j|_{\mathscr{S}_1} = 0$. Approximating the terms about the asymptotic streamlines $\tilde{x}_i$ and neglecting the term $\Delta v_j \Delta n_j$, we find

$$v_j n_j|_{\tilde{\mathscr{S}}_1} = (U\delta_{1j} + \Delta v_j|_{\tilde{x}})(\lambda_j + \Delta n_j|_{\tilde{x}}) \sim U \Delta n_1|_{\tilde{x}} + \lambda_j \Delta v_j \sim 0. \quad (5.18)$$

Thus, from (5.18) $U\Delta n_1 = -\lambda_j \Delta v_j$ and further $x_2^{-\infty} \sim \tilde{\Sigma} \lambda_2$ which when substituted into (5.17) gives

$$\left\{ \frac{-1}{\rho_0} p n_i - u_i u_j n_j \right\} \bigg|_{\tilde{\mathscr{S}}_1} \equiv U(\Delta w_1 \lambda_i - \Delta w_j \lambda_j \delta_{1i}) + U\Omega \tilde{d}_2 \lambda_i - \Omega \tilde{\Sigma} (\Delta v_1 \lambda_2 \lambda_i + \Delta v_{j\neq 1} \lambda_2 \lambda_j \delta_{1i}). \quad (5.19)$$

Substituting $\tilde{x}_i = x_1 \delta_{1i} + \tilde{\Sigma} \lambda_i$ into the asymptotic approximation (2.4*b*) for Δv_j, we find the following asymptotic forms for Δv_1 and $\Delta v_{j\neq 1}$:

$$\Delta v_1 \sim -Uc_1 \left(r^{-3} - 3r^{-5} x_1^2 \right) + Uc_{l\neq 1} (3r^{-5} x_1 \tilde{\Sigma} \lambda_l), \quad (5.20a)$$

$$\Delta v_{j\neq 1} \sim Uc_1 (3r^{-5} x_1 \tilde{\Sigma} \lambda_j) - Uc_{l\neq 1} (\delta_{lj\neq 1} r^{-3} - 3r^{-5} \tilde{\Sigma}^2 \lambda_j \lambda_l). \quad (5.20b)$$

The approximations (5.20) can now be substituted into (5.19), noting that $\lambda_j \lambda_j = 1$, dropping odd powers of λ_j which will integrate to zero on the interval $0 < \lambda < 2\pi$ and dropping odd functions of λ then $\lambda_2 \lambda_j \equiv \delta_{2j}/2$ to give

$$\left\{ \frac{-1}{\rho_0} p n_i - u_i u_j n_j \right\} \bigg|_{\tilde{\mathscr{S}}_1} \equiv U(\Delta w_1 \lambda_i - \Delta w_j \lambda_j \delta_{1i}) + U\Omega \tilde{d}_2 \lambda_i - \tfrac{1}{2} U\Omega \tilde{\Sigma} \left[-c_1 \left(r^{-3} - 3r^{-5} x_1^2 \right) \delta_{2i} - c_{l\neq 1} (\delta_{lj\neq 1} r^{-3} \delta_{2j} - 3r^{-5} \tilde{\Sigma}^2 \delta_{2l}) \delta_{1i} \right]. \quad (5.21)$$

The function $(r^{-3} - 3r^{-5} x_1^2) = \mathrm{d}/\mathrm{d}x_1 (x_1 r^{-3})$ is a perfect derivative of x_1 which when integrated with respect to x_1 will result in terms of the order of $O(\tilde{X}^{-2})$ at each end of $\tilde{\mathscr{S}}_1$. The corresponding term in (5.21) will contribute terms of order $O(a_B^3 U\Omega \tilde{\Sigma}^2 / \tilde{X}^2)$ to the integral over $\tilde{\mathscr{S}}_1$ and can, therefore, be neglected to give

$$\left\{ -\frac{1}{\rho_0} p n_i - u_i u_j n_j \right\} \bigg|_{\tilde{\mathscr{S}}_1} \equiv U(\Delta w_1 \lambda_i - \Delta w_j \lambda_j \delta_{1i}) + U\Omega \tilde{d}_2 \lambda_i + \tfrac{1}{2} U\Omega \tilde{\Sigma} [c_2 (r^{-3} - 3r^{-5} \tilde{\Sigma}^2) \delta_{1i}]. \quad (5.22)$$

The identity (5.22) can now be compared directly with (6.8) of Auton (1987). Note that his (6.8) refers to his integral (6.1) which has the opposite sign to the x_2–component of the force. In Auton's case of a sphere of radius a, as shown in Batchelor (1967, p. 452), $c_l = -a^3\delta_{1l}/2$ and $c_2 = c_3 = 0$. From (5.20*b*), $\Delta v_2|_{\tilde{x}} = Uc_1(3r^{-5}x_1\tilde{\Sigma}\lambda_2) = -Uc_1\tilde{\Sigma}\lambda_2\,\mathrm{d}/\mathrm{d}x_1(r^{-3})$ and, therefore, $\tilde{d}_2 = Uc_1\tilde{\Sigma}\lambda_2 r^{-3}$ which vanishes identically when integrated over $0 < \lambda < 2\pi$. Thus, our expression (5.22) gives for the x_2–component of the integrand of our force integral the identity

$$\left\{\frac{-1}{\rho_0}pn_2 - u_2u_jn_j\right\}\bigg|_{\tilde{\mathscr{S}}_1} \equiv U\Delta w_1\lambda_2. \tag{5.23}$$

Equation (5.23) agrees exactly with (6.8) of Auton, allowing for his difference in sign and that our force integral is divided through by the density.

5.3. *The contribution to the force integral from Δw_i on the disks $\tilde{\mathscr{S}}_0$ and $\tilde{\mathscr{S}}_2$*

We shall first evaluate the integrand of (5.12) on the far upstream disk $\tilde{\mathscr{S}}_0$ where from (1.9) $v_j|_{\tilde{\mathscr{S}}_0} \sim v_j^{-\infty} = U\delta_{1i}$, $\mathrm{w}_j|_{\tilde{\mathscr{S}}_0} \sim w_j^{-\infty} = -\Omega x_2^{-\infty}\delta_{1j}$ and $n_j|_{\tilde{\mathscr{S}}_0} = -\delta_{1j}$ to give

$$\left\{\frac{-1}{\rho_0}pn_i - u_iu_jn_j\right\}|_{\tilde{\mathscr{S}}_0} \equiv -\left\{\left(v_i^{-\infty}w_j^{-\infty} + v_j^{-\infty}w_i^{-\infty}\right)n_j\right\} \equiv -2U\Omega x_2^{-\infty}\delta_{1i}. \tag{5.24}$$

On the far downstream disk, we again have $v_j|_{\tilde{\mathscr{S}}_0} \sim v_j^{+\infty} = U\delta_{1j}$, but the normal vector changes sign, $n_j|_{\tilde{\mathscr{S}}_2} = +\delta_{1j}$, and there is a finite contribution from the rotational disturbance velocity in the trailing vortex, namely $w_j|_{\tilde{\mathscr{S}}_2} \sim w_j^{+\infty} = -\Omega x_2^{+\infty}\delta_{1j} + \Delta w_j^{+\infty} = -\Omega x_2^{-\infty}\delta_{1j} + \Omega D_2\delta_{1j} + \Delta w_j^{+\infty}$ to give

$$\begin{aligned}\left\{\frac{-1}{\rho_0}pn_i - u_iu_jn_j\right\}\bigg|_{\tilde{\mathscr{S}}_2} &\equiv \left\{\left(v_j^{+\infty}w_j^{+\infty} - v_j^{-\infty}w_j^{-\infty}\right)n_i - \left(v_i^{+\infty}w_j^{+\infty} + v_j^{+\infty}w_i^{+\infty}\right)n_j\right\}\\ &\equiv -U\Delta w_i^{+\infty} - U\Omega D_2\delta_{1i} + 2U\Omega x_2^{-\infty}\delta_{1i}.\end{aligned} \tag{5.25}$$

Combining the contributions (5.24) and (5.25) from the two disks we find that the only contribution to the force integral arises from the rotational disturbance velocity in the trailing vortex thus

$$\left\{\frac{-1}{\rho_0}pn_i - u_iu_jn_j\right\}\bigg|_{\tilde{\mathscr{S}}_0+\tilde{\mathscr{S}}_2} \equiv -U\Delta w_i^{+\infty} - U\Omega D_2\delta_{1i}. \tag{5.26}$$

Identity (5.26) can be compared with (6.14) of Auton (1987) for the x_2–component of the force to give

$$\left\{\frac{-1}{\rho_0}pn_2 - u_2u_jn_j\right\}\bigg|_{\tilde{\mathscr{S}}_0+\tilde{\mathscr{S}}_2} \equiv -U\Delta w_2^{+\infty}|_{\tilde{\mathscr{S}}_2}. \tag{5.27}$$

Thus, (5.27) agrees exactly with (6.14) of Auton, taking account of his difference in sign and our division by the fluid density.

6. The lift force on an arbitrarily shaped body

6.1. *The contribution to the force integral from the stream surface $\tilde{\mathscr{S}}_1$*

We shall proceed by evaluating in turn the contributions from the three components $\Delta v_i^{\Omega} + \Delta w_{i(\mathrm{I})}$, $\Delta w_{i(\mathrm{II})}$ and $\Delta w_{i(\mathrm{III})}$ of the rotational disturbance velocity to the term $U(\Delta w_1\lambda_i - \Delta w_j\lambda_j\delta_{1i})$ in (5.22) for the total contribution to the force from $\tilde{\mathscr{S}}_1$. First, consider the contribution from the asymptotic approximation to $\Delta v_i^{\Omega} + \Delta w_{i(\mathrm{I})}$, given by (2.5$b$) and (4.11) as $\Delta v_j^{\Omega} + \Delta w_{j(\mathrm{I})} \sim \Omega c^{\Omega} x_j r^{-3} - \Omega \tilde{d}_2 \delta_{1j} + \Omega \tilde{d}_1 \delta_{2j}$, where $\tilde{d}_1 = -\Delta\varphi|_{\tilde{x}} = c_1 x_1 r^{-3} + c_{l\neq 1} \tilde{\Sigma} r^{-3} \lambda_l$ and $x_j = x_1\delta_{1j} + \tilde{\Sigma}\lambda_j$, which upon substitution, because $\lambda_j\lambda_j = 1$, gives

$$U\big(\big[\Delta v_1^{\Omega} + \Delta w_{1(\mathrm{I})}\big]\lambda_i - \big[\Delta v_j^{\Omega} + \Delta w_{j(\mathrm{I})}\big]\lambda_j\delta_{1i}\big)\big|_{\tilde{\mathscr{S}}_1} = U\Omega c^{\Omega}(x_1 r^{-3}\lambda_i - \tilde{\Sigma} r^{-3}\delta_{1i}) \\ - U\Omega\tilde{d}_2\lambda_i - U\Omega c_1 x_1 r^{-3}\lambda_2\delta_{1i} - U\Omega c_{l\neq 1}\tilde{\Sigma} r^{-3}\lambda_l\lambda_2\delta_{1i}. \quad (6.1)$$

Dropping odd functions of λ that integrate to zero on the interval $0 < \lambda < 2\pi$ then $\lambda_i \equiv 0$, $\lambda_2 \equiv 0$, $\lambda_2\lambda_l \equiv \delta_{2l}/2$ and (6.1) simplifies to

$$U\big(\big[\Delta v_1^{\Omega} + \Delta w_{1(\mathrm{I})}\big]\lambda_i - \big[\Delta v_j^{\Omega} + \Delta w_{j(\mathrm{I})}\big]\lambda_j\delta_{1i}\big)\big|_{\tilde{\mathscr{S}}_1} \\ \equiv -U\Omega\tilde{d}_2\lambda_i - U\Omega\big(c^{\Omega} + \tfrac{1}{2}c_2\big)\tilde{\Sigma} r^{-3}\delta_{1i}. \quad (6.2)$$

Secondly, consider the contribution from $\Delta w_{i(\mathrm{II})}$ whose asymptotic approximation is given by (4.19). By inspection, it can be seen that $\Delta w_{j(\mathrm{II})}\lambda_j \equiv 0$ since $\lambda_j \equiv 0$. In addition dropping odd functions of λ then $\lambda_k\lambda_i \equiv \delta_{ki\neq 1}/2$ we find

$$U(\Delta w_{1(\mathrm{II})}\lambda_i - \Delta w_{j(\mathrm{II})}\lambda_j\delta_{1i})|_{\tilde{\mathscr{S}}_1} \equiv \frac{-1}{8\pi} U\Omega\varepsilon_{1li}\tilde{\Sigma} r^{-3}\int_{\tilde{\mathscr{S}}_0} \partial D_1'/\partial x_3'^{-\infty} x_{l\neq 1}'^{+\infty}\,\mathrm{d}s'^{-\infty}. \quad (6.3)$$

Finally, consider the contribution from $\Delta w_{i(\mathrm{III})}$ whose asymptotic approximation is given by (4.22). Again by inspection it can be seen that $\Delta w_{1(\mathrm{III})}\lambda_i \equiv 0$ because for integration over $0 < \lambda < 2\pi$ then $\lambda_i \equiv 0$ and $\lambda_i\lambda_j\lambda_k \equiv 0$. Moreover, since $\lambda_j\lambda_l \equiv \delta_{jl\neq 1}/2$ then by changing the summation indices to prevent duplication we find

$$U(\Delta w_{1(\mathrm{III})}\lambda_i - \Delta w_{j(\mathrm{III})}\lambda_j\delta_{1i})|_{\tilde{\mathscr{S}}_1} \equiv -\frac{1}{8\pi} U\Omega\delta_{1i}\varepsilon_{lp1}\tilde{\Sigma} r^{-3}\int_{\tilde{\mathscr{S}}_0} \frac{\partial D_{p\neq 1}'}{\partial x_3'^{-\infty}} x_{l\neq 1}'^{+\infty}\,\mathrm{d}s'^{-\infty}. \quad (6.4)$$

We can now combine the three contributions (6.2), (6.3) and (6.4) for $\Delta v_i^{\Omega} + \Delta w_{i(\mathrm{I})}$, $\Delta w_{i(\mathrm{II})}$ and $\Delta w_{i(\mathrm{III})}$, respectively, into the term $U(\Delta w_1\lambda_i - \Delta w_j\lambda_j\delta_{1i})|_{\tilde{\mathscr{S}}_1}$ of (5.22) to give

$$\left\{\frac{-1}{\rho_0} p n_i - u_i u_j n_j\right\}\bigg|_{\tilde{\mathscr{S}}_1} \equiv \frac{-1}{8\pi} U\Omega\tilde{\Sigma} r^{-3}\int_{\tilde{\mathscr{S}}_0}\left[\varepsilon_{1li}\frac{\partial D'_1}{\partial x_3'^{-\infty}} + \delta_{1i}\varepsilon_{lp1}\frac{\partial D'_{p\neq 1}}{\partial x_3'^{-\infty}}\right] x_{l\neq 1}'^{+\infty}\,\mathrm{d}s'^{-\infty} \\ - U\Omega c^{\Omega}\tilde{\Sigma} r^{-3}\delta_{1i} - \tfrac{3}{2}U\Omega c_2\tilde{\Sigma}^3 r^{-5}\delta_{1i}. \quad (6.5)$$

Note that the term $U\Omega\tilde{d}_2\lambda_i$ in the integral for the total contribution (5.22) cancels identically with the contribution from $\Delta w_{1(\mathrm{I})}$ given by (6.2).

Finally, employing the identities

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} r^{-3}\,\mathrm{d}x_1\,\mathrm{d}\lambda = 4\pi\tilde{\Sigma}^{-2}, \qquad \int_0^{2\pi}\int_{-\infty}^{+\infty} r^{-5}\,\mathrm{d}x_1\,\mathrm{d}\lambda = \tfrac{8}{3}\pi\tilde{\Sigma}^{-4},$$

it follows from (6.5) that the contribution $f_i|_{\tilde{\mathscr{S}}_1}$ from $\tilde{\mathscr{S}}_1$ to the total force (5.1) is given by

$$\frac{1}{\rho_0} f_i|_{\tilde{\mathscr{S}}_1} = -\tfrac{1}{2}U\Omega\int_{\mathscr{S}_0}\left[\varepsilon_{1li}\frac{\partial D'_1}{\partial x_3'^{-\infty}} + \delta_{1i}\varepsilon_{lp1}\frac{\partial D'_{p\neq 1}}{\partial x'^{-\infty}_3}\right] x'^{+\infty}_{l\neq 1}\,\mathrm{d}s'^{-\infty} - 4\pi U\Omega(c^{\Omega} + c_2)\delta_{1i}.$$

Identity (6.6) corresponds to our generalization of Auton's equation (6.13), allowing for his sign difference and our placement of the fluid density on the left-hand side of the equation. The proof of the equality between (6.6) and Auton's (6.13) in his case of a spherical body will be addressed in the discussion of § 7.

6.2. *The contribution to the force integral from the disk* $\tilde{\mathscr{S}}_2$

In the same way as in § 6.1 we shall evaluate the contributions to (5.26) from the three components $\Delta w^{+\infty}_{i(\mathrm{I})}$, $\Delta w^{+\infty}_{i(\mathrm{II})}$ and $\Delta w^{+\infty}_{i(\mathrm{III})}$ of the rotational disturbance velocity in the trailing vortex. First, it follows from (4.14) that $\Delta w^{+\infty}_{i(\mathrm{II})}$ is of order $O(a_B^3 \Omega \tilde{X}^{-2})$ on $\tilde{\mathscr{S}}_2$ and therefore contributes a term of order $O(a_B^3 U \Omega \tilde{\Sigma}^2/\tilde{X}^2)$ to the force integral which can be neglected, namely $f_{i(\mathrm{II})}|_{\tilde{\mathscr{S}}_2} = 0$. As argued in § 4.2, $\Delta w^{+\infty}_{i(\mathrm{I})}$ can be approximated by equation (4.11) in the region $\rho^{-\infty} > \Sigma$ whilst it can also be assumed that it is identically zero for $\rho^{-\infty} < \Sigma$. $\Delta w^{+\infty}_{i(\mathrm{III})}$ is approximately equal to the two-dimensional Biot-Savart integral of $\Delta\omega'^{②}_j$ in the region $\rho^{-\infty} < \Sigma$ as derived in (4.24*b*) and given by:

$$\Delta w^{+\infty}_{i(\mathrm{III})} \sim -\frac{1}{2\pi}\Omega\varepsilon_{ijk\neq 1}\int_{\mathscr{S}_2}\frac{\partial D'_j}{\partial x_3'^{-\infty}}\frac{\partial}{\partial x_k}(\log\eta)|_{\tilde{\mathscr{S}}_2}\,\mathrm{d}s'^{+\infty}. \tag{6.7}$$

Note that the two-dimensional Biot-Savart integral is appropriate because $\Delta w^{+\infty}_{i(\mathrm{III})}$ is independent of x_1 in the trailing vortex and could also be obtained from its three-dimensional form (4.20) by employing the argument used in Batchelor (1967, p. 527) to derive his equation (7.3.1).

First, we shall evaluate the contribution of $\Delta w^{+\infty}_{i(\mathrm{I})}$, namely $\Delta w^{+\infty}_{i(\mathrm{I})} = \Omega\varepsilon_{i3k}\tilde{d}_k$, to the force integral on $\tilde{\mathscr{S}}_2$ as given by (5.26). If we approximate the drift vector $\tilde{d}_k|_{\tilde{\mathscr{S}}_2}$ evaluated on $\tilde{\mathscr{S}}_2$ by the total drift $\tilde{D}_k$ we find

$$\frac{1}{\rho_0}f_{i(\mathrm{I})}|_{\tilde{\mathscr{S}}_2} = -U\Omega\varepsilon_{i3k}\int_{\tilde{\mathscr{S}}_2-\mathscr{S}_2}\tilde{d}_k|_{\tilde{\mathscr{S}}_2}\,\mathrm{d}s \sim -U\Omega\varepsilon_{i3k}\int_{\tilde{\mathscr{S}}_2-\mathscr{S}_2}\tilde{D}_k\,\mathrm{d}s. \tag{6.8}$$

Here, $\tilde{D}_k$ on the annulus $\tilde{\mathscr{S}}_2 - \mathscr{S}_2$ is found by taking the limit as $x_1 \to +\infty$ in (2.17) to obtain

$$\tilde{D}_k = \int_{-\infty}^{+\infty} -\Delta\varphi_{,k}|_{\tilde{x}}\,\mathrm{d}x_1 = \int_{-\infty}^{+\infty} -\Delta\varphi_{,k\neq 1}|_{\tilde{x}}\,\mathrm{d}x_1 = \int_{-\infty}^{+\infty} c_{l\neq 1}(\delta_{lk\neq 1}r^{-3} - 3\rho^2 r^{-5}\lambda_l\lambda_k)\,\mathrm{d}x_1. \tag{6.9}$$

Dropping odd functions of λ when $\tilde{D}_k$ is integrated over the interval $0 < \lambda < 2\pi$ then $\lambda_l\lambda_k \equiv \delta_{lk\neq 1}/2$ and, therefore,

$$\tilde{D}_k \equiv c_{k\neq 1}\int_{-\infty}^{+\infty}\left(r^{-3} - \frac{3}{2}\rho^2 r^{-5}\right)\mathrm{d}x_1 \equiv -\tfrac{1}{2}c_{k\neq 1}\int_{-\infty}^{+\infty}\frac{\mathrm{d}}{\mathrm{d}x_1}(x_1 r^{-3})\,\mathrm{d}x_1 = 0. \tag{6.10}$$

The force contribution $f_{i(\mathrm{I})}|_{\tilde{\mathscr{S}}_2}$ due to $\Delta w^{+\infty}_{i(\mathrm{I})}$ from $\tilde{\mathscr{S}}_2$, therefore, is identically zero.

Finally, we evaluate the contribution of $\Delta w^{+\infty}_{i(\mathrm{III})}$ to the force integral on $\tilde{\mathscr{S}}_2$ as given by (5.26). Substituting (6.7) into (5.26) and reversing the order of the double integration we obtain

$$\frac{1}{\rho_0}f_{i(\mathrm{III})}|_{\tilde{\mathscr{S}}_2} = \frac{1}{2\pi}U\Omega\varepsilon_{ijk\neq 1}\int_{\mathscr{S}_2}\frac{\partial D'_j}{\partial x_3'^{-\infty}}\left(\int_{\tilde{\mathscr{S}}_2}\frac{\partial}{\partial x_{k\neq 1}}(\log\eta)|_{\tilde{\mathscr{S}}_2}\,\mathrm{d}s\right)\mathrm{d}s^{+\infty} - U\Omega\delta_{1i}\int_{\mathscr{S}_2}D_2\,\mathrm{d}s. \tag{6.11}$$

Note that by changing the integration variable from $\mathrm{d}s^{+\infty}$ to $\mathrm{d}s^{-\infty}$ using (2.20) then $\int_{\mathscr{S}_2}D_2\,\mathrm{d}s^{+\infty} = \int_{\mathscr{S}_0}D_2\,\mathrm{d}s^{-\infty}$ which is equal to the x_2-component of the drift-volume given by (3.11). Applying the divergence theorem to the inner integral, and the identity for the drift volume given by (3.11) while noting from Batchelor (1967, p. 403) that $c_2 = -\mathscr{V}_B/(4\pi)C_{21}$, then

$$\frac{1}{\rho_0} f_{i(\mathrm{III})}|_{\tilde{\mathscr{S}}_2} = \frac{1}{2\pi} U\Omega\varepsilon_{ijk\neq 1}\int_{\mathscr{S}_2} \frac{\partial D'_j}{\partial x'^{-\infty}_3}\left(\int_0^{2\pi} \log\eta|_{\tilde{\mathscr{C}}_2}\lambda_k\tilde{\Sigma}\,\mathrm{d}\lambda\right)\mathrm{d}\mathscr{s}'^{+\infty} + 2\pi U\Omega c_2\delta_{1i}. \tag{6.12}$$

Here, $\tilde{\mathscr{C}}_2$ is the perimeter contour of $\tilde{\mathscr{S}}_2$. Now because $\rho\sim\tilde{\Sigma}$ on $\tilde{\mathscr{C}}_2$ which is a large distance from the integration domain $\mathscr{S}_2$, we can approximate η to highest order by neglecting terms of order $(\rho'^{+\infty}/\tilde{\Sigma})^2$ to give

$$\begin{aligned}\eta|_{\tilde{\mathscr{C}}_2} &= \left[\left(x_{l\neq 1} - x'^{+\infty}_{l\neq 1}\right)\left(x_{l\neq 1} - x'^{+\infty}_{l\neq 1}\right)\right]^{1/2}\\ &= \left[\tilde{\Sigma}^2 + (\rho'^{+\infty})^2 - 2x'^{+\infty}_{l\neq 1}x_{l\neq 1}\right]^{1/2} \sim \tilde{\Sigma}\left(1 - \frac{\rho'^{+\infty}}{\tilde{\Sigma}}\lambda'_l\lambda_l\right).\end{aligned} \tag{6.13}$$

Since for small x then $\log(1+x)\sim x$, it follows that on $\tilde{\mathscr{C}}_2$

$$\log\eta|_{\tilde{\mathscr{C}}_2} \sim \log\tilde{\Sigma} - \rho'^{+\infty}/\tilde{\Sigma}\lambda'_l\lambda_l. \tag{6.14}$$

We can now substitute the approximation (6.14) for log η into (6.12), noting that in the integration with respect to λ over the interval $0<\lambda<2\pi$ then $\lambda_l\equiv 0$ and $\lambda_l\lambda_k\equiv\delta_{lk\neq 1}/2$ to obtain

$$\int_0^{2\pi}\log\eta|_{\tilde{\mathscr{C}}_2}\lambda_k\tilde{\Sigma}\mathrm{d}\lambda \sim -\pi\rho'^{+\infty}\lambda'_k = -\pi x'^{+\infty}_{k\neq 1}. \tag{6.15}$$

Now substituting (6.15) into (6.12), we obtain the required identity for $f_{i(\mathrm{III})}|_{\tilde{\mathscr{S}}_2}$, the contribution from $\Delta w^{+\infty}_{i(\mathrm{III})}$ to the force from $\tilde{\mathscr{S}}_2$, as

$$\frac{1}{\rho_0} f_{i(\mathrm{III})}|_{\tilde{\mathscr{S}}_2} = -\tfrac{1}{2}U\Omega\varepsilon_{ijk\neq 1}\int_{\mathscr{S}_2}\frac{\partial D'_j}{\partial x^{-\infty}_3}x'^{+\infty}_k\,\mathrm{d}\mathscr{s}'^{+\infty} + 2\pi U\Omega c_2\delta_{1i}. \tag{6.16}$$

Combining (6.8), (6.10) and (6.16), changing the surface integration in (6.16) from $\mathrm{d}\mathscr{s}'^{+\infty}$ to $\mathrm{d}\mathscr{s}'^{-\infty}$ using (2.20) and writing $\varepsilon_{ijl\neq 1} = \varepsilon_{1jl}\delta_{1i} + \varepsilon_{i1l}\delta_{1j}$, we find the contribution $f_i|_{\tilde{\mathscr{S}}_2}$ to the total force from $\tilde{\mathscr{S}}_2$ is given by

$$\frac{1}{\rho_0} f_i|_{\tilde{\mathscr{S}}_2} = -\tfrac{1}{2}U\Omega\int_{\mathscr{S}_0}\left[\varepsilon_{i1l}\frac{\partial D'_1}{\partial x'^{-\infty}_3} + \delta_{1i}\varepsilon_{1jl}\frac{\partial D'_{j\neq 1}}{\partial x'^{-\infty}_3}\right]x'^{+\infty}_l\,\mathrm{d}\mathscr{s}'^{-\infty} + 2\pi U\Omega c_2\delta_{1i}. \tag{6.17}$$

Finally, by writing $\varepsilon_{i1l} = \varepsilon_{1li}$ and $\varepsilon_{1jl} = -\varepsilon_{lj1}$, we can alternatively express (6.17) as below, this form now showing explicitly the components of force parallel (δ_{1i}) and perpendicular (ε_{1li}) to the ambient flow direction.

$$\frac{1}{\rho_0} f_{i(\mathrm{III})}|_{\tilde{\mathscr{S}}_2} = -\tfrac{1}{2}U\Omega\int_{\mathscr{S}_0}\left[\varepsilon_{1li}\frac{\partial D'_1}{\partial x'^{-\infty}_3} - \delta_{1i}\varepsilon_{lj1}\frac{\partial D'_{j\neq 1}}{\partial x'^{-\infty}_3}\right]x'^{+\infty}_l\,\mathrm{d}\mathscr{s}'^{-\infty} + 2\pi U\Omega c_2\delta_{1i}. \tag{6.18}$$

Identity (6.18) corresponds to our generalization of Auton's equation (6.15) whose equality will be proved in § 7 for the case of the sphere.

6.3. *The summed contributions to the force integral from $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$*

The total force f_i on the body is obtained by adding the contributions $f_i|_{\tilde{\mathscr{S}}_1}$ given by (6.6) and $f_i|_{\tilde{\mathscr{S}}_2}$ given by (6.18). Note that the terms parallel to the flow direction, that involve the off-axial components $D_{j\neq 1}$ of the total drift vector, cancel exactly, leaving the axial force proportional to $c^{\Omega} + c_2/2$. We can evaluate c^{Ω} by taking the sum of (5.22) and (5.26) to show the axial force is given by

$$f_1 = -U\left[\int_{\tilde{\mathscr{S}}_1}\left\{\Delta w_j\lambda_j - \tfrac{1}{2}\Omega\tilde{\Sigma}[c_2(r^{-3} - 3r^{-5}\tilde{\Sigma}^2)\right\}\,\mathrm{d}\mathscr{s} + \int_{\tilde{\mathscr{S}}_2}\left\{\Delta w_1^{+\infty} + \Omega D_2\right\}\mathrm{d}\mathscr{s}\right].$$

Furthermore, by applying the integral identities used to derive (6.6) from (6.5) and to derive (6.12) from (6.11) it follows that

$$f_1 = -U\left[\int_{\tilde{\mathscr{S}}_1}\Delta w_j\lambda_j\,\mathrm{d}\mathscr{s} + \int_{\tilde{\mathscr{S}}_2}\Delta w_1^{+\infty}\,\mathrm{d}\mathscr{s}\right].$$

Finally, by applying the bounds involving Δw_j and Δn_j that were used to derive (5.17) from (5.16), we also find that $f_1 \sim -U\int_{\tilde{\mathscr{S}}}\Delta w_j n_j\,\mathrm{d}\mathscr{s}$. Thus, the axial force is proportional to the volume flux generated by the disturbance velocity across the asymptotic surface $\tilde{\mathscr{S}}$ which, because the velocity field is incompressible, must be equal to the volume flux across the body surface $\mathscr{S}_B$, namely $f_1 = -U\int_{\mathscr{S}_B}\Delta w_j n_j\,\mathrm{d}\mathscr{s}$. However, because $w_j n_j|_B = 0$ then $f_1 = U\Omega\int_{\mathscr{S}_B} x_2 n_1\,\mathrm{d}\mathscr{s}$ and, by applying the divergence theorem, $f_1 = 0$ and, therefore, $c^{\Omega} = -c_2/2$. The total force, therefore, is given by (6.19) which acts in a direction perpendicular to the flow direction

$$\frac{1}{\rho_0}f_i = \frac{1}{\rho_0}f_i|_{\tilde{\mathscr{S}}_1+\tilde{\mathscr{S}}_2} = -U\Omega\varepsilon_{1li}\int_{\mathscr{S}_0}\frac{\partial D'_1}{\partial x_3'^{-\infty}}x_{l\neq1}'^{+\infty}\,\mathrm{d}\mathscr{s}'^{-\infty}. \tag{6.19}$$

The identity (6.19) corresponds to our generalization of Auton's equation (6.16). If comparing the two identities, note the misprinted sign in Auton's paper. The equality of (6.19) with Auton's (6.16) for his case of a spherical body is left to the discussion in § 7.

One final step is required in our argument to express the total force, (6.19), in a form that is more readily comparable to the result for the sphere. To do this, we express the off-axis coordinates $x_{l\neq1}'^{+\infty}$ of the far downstream streamlines in terms of the total drift as

$$x_{l\neq1}'^{+\infty} = x_{l\neq1}'^{-\infty} - D'_{l\neq1}. \tag{6.20}$$

Now substitute the identity $(\partial/\partial x_3'^{-\infty}(D'_1 x_{l\neq1}'^{-\infty}) = D'_1\delta_{3l} + (\partial D'_1/\partial x_3'^{-\infty})x_{l\neq1}'^{-\infty}$ into (6.19). Note that when evaluated on the perimeter contour $\mathscr{C}_0$ of $\mathscr{S}_0$, D'_1 can be approximated by $-\Delta\varphi'$ and, therefore, $D'_1 x_{l\neq1}'^{-\infty}$ is of order $O(a_B^3\tilde{\Sigma}/\tilde{X}^2)$ on $\mathscr{C}_0$ and makes a contribution of $O(a_B^3 U\Omega\tilde{\Sigma}^2/\tilde{X}^2)$ to the force integral which can be neglected. Since $\varepsilon_{1li}\delta_{3l} = -\delta_{2i}$ and $\varepsilon_{1li} = -\varepsilon_{il1}$, we can alternatively write

$$\frac{1}{\rho_0}f_i = -U\Omega\delta_{2i}\int_{\mathscr{S}_0}D'_1\,\mathrm{d}\mathscr{s}' - U\Omega\varepsilon_{il1}\int_{\mathscr{S}_0}\frac{\partial D'_1}{\partial x_3'^{-\infty}}D'_{l\neq1}\,\mathrm{d}\mathscr{s}'^{-\infty}. \tag{6.21}$$

Finally, applying Darwin's theorem in the form (3.11), we can now relate the force to the added mass coefficient C_{11} and the volume of the body $\mathscr{V}_B$ as

$$\frac{1}{\rho_0}f_i = -\mathscr{V}_B C_{11}U\Omega\delta_{2i} - U\Omega\varepsilon_{il1}\int_{\mathscr{S}_0}\frac{\partial D'_1}{\partial x_3'^{-\infty}}D'_l\,\mathrm{d}\mathscr{s}'^{-\infty}. \tag{6.22}$$

7. Discussion

First, we will make some general observations about the physical origin of the newly identified lift term f_i given by (7.1), below, then we explain how the result can be applied to determining the force on bodies in real fluids taking into account the effects of boundary-layer vorticity and, finally, we discuss the body shapes for which the term is non-zero.

$$\frac{1}{\rho_0}f_i = -U\Omega\varepsilon_{il1}\int_{\mathscr{S}_0}\frac{\partial D_1'}{\partial x_3'^{-\infty}}\,D_l'\,\mathrm{d}s'^{-\infty}. \tag{7.1}$$

Since the integrand of (7.1) is of order $O(a_B^6\rho^{-5})$ as $\rho\to+\infty$, then the physical effect from which the lift force derives must lie very close to the body surface and, therefore, must be sensitive to small changes in the body shape.

When applying the results to real fluids it is important to appreciate that the surface $\mathscr{S}_B$, being arbitrary, need not be coincident with the body. $\mathscr{S}_B$ could instead be defined as the fluid surface $\mathscr{S}_{BF}$ that encloses the fluid immediately surrounding the body that is affected by boundary-layer or shear-layer vorticity. Necessarily, $\mathscr{S}_{BF}$ must be closed and, therefore, this approach cannot be used on a body with a turbulent wake when the bow streamlines do not close behind the stern. On the upstream bow surface, unless it has a re-entrant shape, $\mathscr{S}_{BF}$ would lie close to the body being separated only by a narrow boundary-layer region. If the bow surface is re-entrant, however, a stagnation or recirculation zone may form in which case $\mathscr{S}_{BF}$ should be chosen to be coincident with the outer boundary of this bow fluid region. If the body has a slender shape, whose chord is aligned to the mean flow direction, then separation may not occur and $\mathscr{S}_{BF}$ would differ only from the body shape by the intermediate boundary layer. Bodies with bluff or re-entrant stern shapes are likely to cause separation and, in a similar way to that described for the re-entrant bow shape, $\mathscr{S}_{BF}$ should be chosen as the outer boundary of the wake recirculation zone whose dimension may be comparable with the transverse cross-section of the body. With our results for the force on $\mathscr{S}_{BF}$, the net force on the body can then be calculated by combining it with an additional force balance for the intermediate fluid region $\mathscr{S}_{BF}-\mathscr{S}_B$. For boundary-layer regions, the tangential shear stress would be predominant. For wake recirculation zones the axial drag force would be predominant. Note that the axial force is predicted to be identically zero under the ideal-fluid assumptions of our theoretical analysis. Note also that any substantial difference in size between $\mathscr{S}_B$ and $\mathscr{S}_{BF}$ would give rise to significant differences between the added mass tensor coefficients C_{ij}^B and total drift-vector D_l^B for the body $\mathscr{S}_B$ and the corresponding quantities C_{ij}^{BF} and D_l^{BF} for the enclosing region $\mathscr{S}_{BF}$.

Clearly, for body shapes for which the off-axis total drift vector components are zero, namely $D_{l\neq 1}=0$, then (7.1) is identically zero. Such shapes include bodies of revolution whose symmetry axis is aligned with the ambient flow. They also include bodies that have reflective symmetry about three mutually orthogonal planes with one of the principal axes aligned with the flow direction. This latter class include the ellipsoid. It is possible to argue that the off-axis total drift vector components are identically zero for these shapes on geometrical grounds by considering the drift experienced by a string of particles that initially lie on a far upstream circle $\rho=\rho^{-\infty}$. For all of these body shapes, their symmetry demands that the particle string when on the far downstream side of the body remains circular with radius $\rho=\rho^{+\infty}$, say. In addition, since the string maps out the surface of a stream tube and the axial component of the velocity field is equal to U at both the far upstream and far downstream ends of the stream tube, then the volume fluxes through both circles are equal. The radii of the two circles are, therefore, also equal, namely $\rho^{-\infty}=\rho^{+\infty}$. It follows that the off-axis starting coordinates of the string particles are equal to their off-axis finishing coordinates, namely $x_{l\neq 1}^{-\infty} = x_{l\neq 1}^{+\infty}$ and, therefore, $D_{l\neq 1} = 0$.

The force term (7.1) will also be identically zero for body shapes that generate a total drift function that is symmetrical about two mutually orthogonal lines in the far downstream $x_2 \times x_3$ plane. The reason is simply that such a shape would result in $D_{l\neq 1}$ being symmetric about these two lines while $\partial D_1/\partial x_3^{-\infty}$ would be antisymmetric about the same lines. This follows because the differential $\partial x_3^{-\infty}$ can be resolved into two mutually perpendicular differentials along the two symmetry lines. Such body shapes include ellipsoids irrespective of the how the ellipsoid is oriented to the flow direction. Another that has been considered in detail by the author is the binary sphere system. For this latter case the additional force term is identically zero irrespective of how the line of centres of the two spheres is aligned to the flow.

In all these exceptional cases, the lift force given by (6.22) reduces to

$$\frac{1}{\rho_0} f_i = -\mathscr{V}_B C_{11} U \Omega \delta_{2i}. \tag{7.2}$$

Equation (7.2) is in agreement with the analysis of Auton (1987) for the sphere and also the combined experimental and computational fluid dynamic studies reported by Rife *et al.* (1997) for some bodies of revolution.

The following discussion will address the three main areas of the proof. First, the generalization of Darwin's theorem in § 3. Secondly, the derivation of the asymptotic approximations to the rotational disturbance velocity given in § 4. Finally, the derivation of the total force from the asymptotic surface integral in §§ 5 and 6. Detailed comparisons will be provided with the independent studies by Darwin (1953), Lighthill (1956, 1957) and Auton (1987) in order to provide support for the correctness of our proof.

Throughout the whole of our argument we make only two applications of Darwin's theorem. First, at the beginning of § 6.3 when we argue that the axial force f_1 is identically zero. Secondly, when we relate the final expression (6.21) for the lift force to the added mass coefficient C_{11}. Consequently, to determine the off-axial lift force we use only the x_1-component of the general identity that we derived in (3.11) and which Darwin originally determined in his (8.9), his hydrodynamic mass H being equal to our added mass $\mathscr{V}_B C_{11}$.

We shall now compare in detail our analysis in § 4 with that of § 3 in Lighthill (1956). It is important to note that Lighthill's analysis supposes that the only non-zero component of the disturbance vorticity (referred to as the vorticity change by Lighthill) in the trailing vortex lies in the x_1-direction. This assumption is embodied in his equation (15). In our general analysis, however, the disturbance vorticity $\Delta\omega_i^{+\infty}$ in the trailing vortex is shown to have the form given by (1.16) as

$$\Delta\omega_l^{+\infty} = -\Omega \frac{\partial D_l}{\partial x_3^{-\infty}}. \tag{7.3}$$

Thus, Lighthill's analysis supposes that both off-axis components of the total drift vector have zero gradients in the x_3-direction, namely $\partial D_{l\neq 1}/\partial x_3^{-\infty} = 0$. However, because the total drift vector D_l tends to zero as $\rho^{-\infty} \to +\infty$, then the off-axis total drift components must be identically zero, namely $D_{l\neq 1} = 0$. Lighthill's argument, therefore, is restricted to a limited class of body shapes some of which are discussed earlier in this section. It follows from our identity (3.11), that Lighthill's added mass coefficient tensor has the form $C_{11}\delta_{1i}$ and, therefore, the coefficients c_2 and c_3 of the asymptotic form of the irrotational disturbance velocity Δv_i are identically zero. This is also the reason why, in Lighthill's analysis of the asymptotic form for Δw_i, the irrotational velocity Δv_i^{Ω} can be neglected since it is of order $O(r^{-3})$.

We shall now compare our general results for the asymptotic approximations of the rotational disturbance velocity with those of Lighthill, noting that in his notation $\Omega = -A$. The following identities (7.4) hold, therefore, for the recurrent terms in our equations where in his case of a sphere of radius a then $\mathscr{V}_B C_{11} = \mathscr{V}_B C_M = 2\pi a^3/3$.

$$\int_{\mathscr{S}_0} \frac{\partial D'_{p\neq 1}}{\partial x'^{-\infty}_3} x'^{+\infty}_{l\neq 1}\, \mathrm{d}s'^{-\infty} = 0, \qquad \int_{\mathscr{S}_0} \frac{\partial D'_1}{\partial x'^{-\infty}_3} x'^{+\infty}_l\, \mathrm{d}s'^{-\infty} = -\mathscr{V}_B C_M \delta_{3l}. \tag{7.4}$$

Our result (4.11) for $\Delta\tilde{w}_i$, the asymptotic form of the rotational disturbance velocity $\Delta w_{i(\mathrm{I})}$ on the streamlines that remain far from the body, agrees exactly with (19) of Lighthill (1956). Note that he describes $\Delta\tilde{w}_i$ as the velocity field corresponding to his asymptotic form (18) of his vorticity change $\boldsymbol{\omega}_1$. Our result (4.18) for $\Delta w_{i(\mathrm{II})}$ is equal to $\boldsymbol{v}_2$ of Lighthill's equation (20) where his $\boldsymbol{\omega}_2$, the difference in the vorticity change $\boldsymbol{\omega}_1$ from his asymptotic form (18), is equal to our $\Delta\omega_j^{②}$. Thus, our (4.18) is in agreement with Lighthill's (20), noting that the sign of Lighthill's (20) is incorrect as pointed out in Lighthill (1957). Substituting (7.4) into (4.18) we obtain the asymptotic form (7.5) below for $\Delta w_{i(\mathrm{II})}$, which is in agreement with Lighthill's (22) once the error in the sign of Lighthill's equation is corrected.

$$\Delta w_{i(\mathrm{II})} \sim \frac{-1}{4\pi} \mathscr{V}_B C_M \Omega \varepsilon_{i3k} (r^{-1})_{,k}. \tag{7.5}$$

Lighthill (1956) makes the mistake of not calculating the contribution from $\Delta w_{i(\mathrm{III})}$, which he corrects in Lighthill (1957). Quoting from the theory of the horseshoe vortex, his revised expression for $\mathbf{v}_2$ is given by his (85), which now corresponds to our sum $\Delta w_{i(\mathrm{II})} + \Delta w_{i(\mathrm{III})}$. To obtain an expression for $\Delta w_{i(\mathrm{III})}$ substitute (7.4) above into (4.22) to give

$$\Delta w_{i(\mathrm{III})} \sim \frac{1}{4\pi} \mathscr{V}_B C_M \Omega \varepsilon_{i1k} (\rho^{-2}[1 + x_1 r^{-1}]\{\delta_{k3} - 2\lambda_k\lambda_3\} - x_1 r^{-3}\lambda_k\lambda_3). \tag{7.6}$$

If we now write $\varepsilon_{i3k} = -\delta_{1i}\delta_{2k} + \delta_{2i}\delta_{1k}$ and $(r^{-1})_{,k} = -x_1 r^{-3}\delta_{1k} - \rho r^{-3}\lambda_k$, we can split (7.5) for $\Delta w_{i(\mathrm{II})}$ into δ_{1i}, δ_{2i} and δ_{3i} components using the identities $\varepsilon_{i3k}(r^{-1})_{,k} = \rho r^{-3}\lambda_2\delta_{1i} - x_1 r^{-3}\delta_{2i}$ and $\varepsilon_{i1k} = \delta_{3i}\delta_{2k} - \delta_{2i}\delta_{3k}$. Combined with the identities $1 - \lambda_3^2 = \lambda_2^2$ and $1 - 2\lambda_3^2 = -1 + 2\lambda_2^2$, we find

$$\Delta w_{i(\mathrm{II})} + \Delta w_{i(\mathrm{III})} \sim \frac{-1}{4\pi} \mathscr{V}_B C_M \Omega \Theta_i \tag{7.7}$$

where the vector Θ_i can be expressed in terms of its three Cartesian components as

$$\Theta_i = \rho r^{-3}\lambda_2\delta_{1i} + \rho^{-2}[1 + x_1 r^{-1}]\delta_{2i} - \{2\rho^{-2}[1 + x_1 r^{-1}] + x_1 r^{-3}\}\lambda_2^2\delta_{2i} \\ - \{2\rho^{-2}[1 + x_1 r^{-1}] + x_1 r^{-3}\}\lambda_2\lambda_3\delta_{3i}. \tag{7.8}$$

The equality between our (7.7) and (85) of Lighthill (1957) follows by noting that

$$\Theta_i = \{x_2\rho^{-2}[1 + x_1 r^{-1}]\}_{,i}. \tag{7.9}$$

The identity (7.9) can be proved simply by writing $x_2\rho^{-2}[1 + x_1 r^{-1}]$ in cylindrical coordinates as $\rho^{-1}\lambda_2[1 + x_1 r^{-1}]$ and employing the partial derivatives $(\partial/\partial x_1)(.)$ and $(\partial/\partial x_{k\neq 1})(.) = \lambda_k(\partial/\partial\rho)(.) - \lambda_3/\rho\delta_{2k}(\partial/\partial\lambda)(.) + \lambda_2/\rho\delta_{3k}(\partial/\partial\lambda)(.)$ whilst noting that $\partial\lambda_2/\partial\lambda = -\lambda_3$, $\lambda_2^2 + \lambda_3^2 = 1$ and $\partial r/\partial\rho = \rho/r$.

We shall now compare our evaluation of the force integral (5.1) with that in §6 of Auton (1987) for the sphere. The contribution $f_i|_{\tilde{\mathscr{S}}_1}$ to the force from the stream surface $\tilde{\mathscr{S}}_1$ is given by (6.6) which upon substituting the identities (7.4) for the sphere gives

$$\frac{1}{\rho_0} f_i|_{\tilde{\mathscr{S}}_1} = \tfrac{1}{2}\mathscr{V}_B C_M U \Omega \varepsilon_{13i} = -\tfrac{1}{3}\pi a^3 U \Omega \delta_{2i}. \tag{7.10}$$

This result agrees with Auton's (6.13) which in our notation is equal to $-f_i|_{\tilde{\mathscr{S}}_1}$. Similarly, (6.18) yields the force contribution $f_i|_{\tilde{\mathscr{S}}_2}$ from the downstream disk as

$$\frac{1}{\rho_0} f_i|_{\tilde{\mathscr{S}}2} = \tfrac{1}{2}\mathscr{V}_B C_M U \Omega \varepsilon_{13i} = -\tfrac{1}{3}\pi a^3 U \Omega \delta_{2i}. \tag{7.11}$$

Identity (7.11), therefore, is in agreement with Auton's (6.15) which in our notation is equal to $-f_i|_{\tilde{\mathscr{S}}2}$. Thus, $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$ contribute equally to the total force in Auton's argument. In our argument the contributions from $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$, given by (6.6) and (6.18), respectively, differ, but only in the x_1-component of the force which vanishishes identically when the two contributions are added together. The result is that $\tilde{\mathscr{S}}_1$ and $\tilde{\mathscr{S}}_2$, as in Auton's case, contribute equally to the net force each contributing

$$-\tfrac{1}{2} U \Omega \varepsilon_{1li} \int_{\mathscr{S}_0} \frac{\partial D_1'}{\partial x_3'^{-\infty}} x_{l\neq 1}'^{+\infty}\, \mathrm{d}s'^{-\infty}.$$

It is important to point out that our derivation in §6.2 for the contribution to the force differs from Auton's in a subtle but important way. Auton employs his identity (6.6) for the rotational disturbance velocity in the trailing vortex. In our argument, we employ the two-dimensional Biot-Savart integral given by (7.12), namely

$$\Delta w_{i(\mathrm{III})}^{+\infty} \sim \frac{-1}{2\pi} \Omega \varepsilon_{ijk\neq 1} \int_{\mathscr{S}_2} \frac{\partial D_j'}{\partial x_3'^{-\infty}} \frac{\partial}{\partial x_k} (\log \eta)|_{\tilde{\mathscr{S}}_2}\, \mathrm{d}s'^{+\infty}. \tag{7.12}$$

In fact, Auton's (6.6) is equal to our (7.12) provided the integration domain $\mathscr{S}_2$ in (7.12) is replaced by $\tilde{\mathscr{S}}_2$. Note that in Auton's notation his Σ corresponds to our $\tilde{\Sigma}$. The difference is that the integration region in our Biot-Savart integral (7.2) is over $\mathscr{S}_2$ and not $\tilde{\mathscr{S}}_2$. We purposely chose the radius of $\mathscr{S}_2$ as Σ where $\Sigma \ll \tilde{\Sigma}$. This crucial step then allows us to evaluate the force integral (6.11), having first applied the divergence theorem, by using an asymptotic approximation of the integrand on the perimeter contour of $\tilde{\mathscr{S}}_2$. This step is only possible because the points on the perimeter contour of $\tilde{\mathscr{S}}_2$ lie at a large distance from the integration range $\mathscr{S}_2$.

Appendix A. $\tilde{d}_i = O(a_B^3 \tilde{\Sigma}^{-2})$ and $\Delta n_i = O(a_B^3 \tilde{\Sigma}^{-3})$

The approximation for $\tilde{d}_i$ follows immediately by noting that $\Delta\varphi_{,i} = O(a_B^3 r^{-3})$ where $(r^2)|_{\tilde{x}} = x_1^2 + \tilde{\Sigma}^2$ and, therefore,

$$\tilde{d}_i = \int_{-\infty}^{x_1} -\Delta\varphi_{,i}|_{\tilde{x}}\, \mathrm{d}x_1 = O\left(a_B^3 \int_{-\infty}^{+\infty} r^{-3}\, \mathrm{d}x_1\right) = O\left(a_B^3 \tilde{\Sigma}^{-2}\right). \tag{A 1}$$

To approximate $\Delta n_i|_{\tilde{x}}$ we make use of the following identity for the normal n_i which is obtained by parameterization of the surface with respect to the coordinates λ and x_1, for example as shown in Pozrikidis (1997, pp. 15–16).

$$n_i \mathrm{d}s = \varepsilon_{ijk} \left.\frac{\partial x_j}{\partial \lambda}\right|_{\tilde{x}} \left.\frac{\partial x_k}{\partial x_1}\right|_{\tilde{x}} \mathrm{d}\lambda\, \mathrm{d}x_1 = (\lambda_i + \Delta n_i|_{\tilde{x}})\, \tilde{\Sigma}\, \mathrm{d}\lambda\, \mathrm{d}x_1. \tag{A 2}$$

Note that, to highest order, the stream surface is approximated by the cylinder $\rho = \tilde{\Sigma}$ whose normal is λ_i. To second order, the surface vector is given by (2.16) as

$$x_i = \tilde{x}_i - \tilde{d}_i = x_1\delta_{1i} + \tilde{\Sigma}\lambda_i - \tilde{d}_i. \tag{A 3}$$

Substituting (A 3) into the partial differentials of (A 2) we find

$$\partial x_j/\partial\lambda|_{\tilde{x}} \sim \tilde{\Sigma}\frac{\partial\lambda_j}{\partial\lambda} - \frac{\partial\tilde{d}_j}{\partial\lambda}, \quad \left.\frac{\partial x_k}{\partial x_1}\right|_{\tilde{x}} \sim \delta_{1k} - \left.\frac{\partial\tilde{d}_k}{\partial x_1}\right|_{\tilde{x}} = \delta_{1k} + \Delta\varphi_{,k}|_{\tilde{x}} \tag{A 4}$$

Now substitute (A 4) into the identity (A 2) for the normal vector to obtain

$$(\lambda_i + \Delta n_i|_{\tilde{x}}) = \varepsilon_{ijk}(\delta_{1k} + \Delta\varphi_{,k})\,\tilde{\Sigma}^{-1}\frac{\partial x_j}{\partial\lambda} = \varepsilon_{ij1}\frac{\partial\lambda_j}{\partial\lambda} - \varepsilon_{ij1}\,\tilde{\Sigma}^{-1}\frac{\partial\tilde{d}_j}{\partial\lambda} + \varepsilon_{ijk}\frac{\partial\lambda_j}{\partial\lambda}\Delta\varphi_{,k} + O(\tilde{\Sigma}^{-2}r^{-3}). \tag{A 5}$$

Finally, noting that $\varepsilon_{ij1}\partial\lambda_j/\partial\lambda = \lambda_i$ and $\Delta\varphi_{,k} = O(a_B^3 r^{-3})$, we obtain the required approximation for $\Delta n_i|_{\tilde{x}}$, namely

$$\Delta n_i|_{\tilde{x}} \sim -\varepsilon_{ij1}\tilde{\Sigma}^{-1}\frac{\partial\tilde{d}_j}{\partial\lambda} + \varepsilon_{ijk}\frac{\partial\lambda_j}{\partial\lambda}\Delta\varphi_{,k} = O\left(a_B^3\tilde{\Sigma}^{-3}\right). \tag{A 6}$$

Appendix B. Asymptotic identities for $\boldsymbol{B}_k = \int_X^{+\infty}(\partial/\partial x_k)(1/\xi)\mathrm{d}x_1'$

Consider the cases $k = 1$ and $k \neq 1$ separately. Noting that $\xi = [(x_1' - x_1)^2 + \eta^2]^{1/2}$ we integrate with respect to x_1' to obtain

$$\boldsymbol{B}_{k=1} = \left[-[(x_1' - x_1)^2 + \eta^2]^{-1/2}\right]_X^{+\infty} = [(x_1 - X)^2 + \eta^2]^{-1/2}, \tag{B 1}$$

$$\boldsymbol{B}_{k\neq1} = \int_X^{+\infty}(x_{k\neq1}' - x_{k\neq1})\,\xi^{-3}\mathrm{d}x_1' = (x_{k\neq1}' - x_{k\neq1})\,\eta^{-2}[(x_1' - x_1)[(x_1' - x_1)^2 + \eta^2]^{-1/2}]_X^{+\infty}, \tag{B 2a}$$

$$= (x_{k\neq1}' - x_{k\neq1})\,\eta^{-2}\left[1 + (x_1 - X)[(x_1 - X)^2 + \eta^2]^{-1/2}\right]. \tag{B 2b}$$

We need to consider the evaluations of these functions at x_k which lie on the stream surface $\tilde{\mathscr{S}}_1$ with radius $\tilde{\Sigma}$ and length $\tilde{X}$. The integration variable x_k' on the other hand lies within the region $\mathscr{V}$ which has radius Σ and length X. Now since we have assumed that $a_B \ll \Sigma \ll X \ll \tilde{\Sigma} \ll \tilde{X}$ we can then make the approximation $(x_1 - X) \sim x_1$ and neglect terms of order ρ'^2/ρ^2 to obtain

$$\eta^2 = (x_{l\neq1} - x_{l\neq1}')(x_{l\neq1} - x_{l\neq1}') = \rho^2 + \rho'^2 - 2x_{l\neq1}'x_{l\neq1} \sim \rho^2 - 2x_{l\neq1}'x_{l\neq1}. \tag{B 3a}$$

Furthermore, by the binomial theorem

$$\eta^{-2} \sim \rho^{-2}(1 + 2\rho^{-2}x_{l\neq1}'x_{l\neq1}). \tag{B 3b}$$

Noting that $r^2 = x_1^2 + \rho^2$, it follows from (B 1) that

$$\boldsymbol{B}_{k=1} \sim [x_1^2 + \eta^2]^{-1/2} \sim [x_1^2 + \rho^2 - 2x_{l\neq1}'x_{l\neq1}]^{-1/2} \sim r^{-1}(1 + r^{-2}x_{l\neq1}'x_{l\neq1}) \sim r^{-1} + (\rho r^{-3}\lambda_l)x_{l\neq1}', \tag{B 4}$$

and from (B 2*b*) that

$$\boldsymbol{B}_{k\neq1} \sim (x'_{k\neq1} - x_{k\neq1})\eta^{-2}\left[1 + x_1\left[x_1^2 + \eta^2\right]^{-1/2}\right]$$
$$\sim \rho^{-2}(x'_{k\neq1} - x_{k\neq1})(1 + 2\rho^{-2}x'_{l\neq1}x_{l\neq1})[[1 + x_1 r^{-1}] + x_1 r^{-3}x'_{l\neq1}x_{l\neq1}]. \quad \text{(B 5}a\text{)}$$

Neglecting terms of order ρ'^2/ρ^2 in $\boldsymbol{B}_{k\neq1}$, we find

$$\boldsymbol{B}_{k\neq1} \sim \rho^{-2}[1 + x_1 r^{-1}]\{(x'_{k\neq1} - x_{k\neq1}) - 2\rho^{-2}\, x_{k\neq1}x_{l\neq1}x'_{l\neq1}\}$$
$$- x_1 r^{-3}\rho^{-2}\, x_{k\neq1}x_{l\neq1}x'_{l\neq1}. \quad \text{(B 5}b\text{)}$$

By writing $x_{k\neq1} = \rho\lambda_k$, (B 5*b*) can be alternatively expressed as

$$\boldsymbol{B}_{k\neq1} \sim -\rho^{-1}[1 + x_1 r^{-1}]\lambda_k + (\rho^{-2}[1 + x_1 r^{-1}]\{\delta_{k\neq1l} - 2\lambda_k\lambda_l\} - x_1 r^{-3}\lambda_k\lambda_l)x'_{l\neq1}. \quad \text{(B 5}c\text{)}$$

Appendix C. $\Delta\omega_i - \Delta\omega_i^{+\infty} \sim O(\Omega a_B^3 x_1^{-3})$ as $x_1 \to +\infty$

By the definition given in equation (4.7), it follows that for large positive values of x_1, where the streamlines asymptotically approach $\tilde{\mathbf{x}}^+$ and the evaluation point is so far from the body that the asymptotic form for the disturbance velocity $\Delta\varphi_{,i}$ can be used in the integral, we find

$$\Delta\omega_i - \Delta\omega_i^{+\infty} \sim \Omega\frac{\partial}{\partial x_3^{-\infty}}\left[\int_{x_1}^{+\infty} \Delta\varphi_{,i}|_{\tilde{x}^+}\,\mathrm{d}x_1\right] \sim -\Omega\frac{\partial}{\partial x_3^{-\infty}}\left[\int_{x_1}^{+\infty} (c_l x_l r^{-3})_{,i}|_{\tilde{x}^+}\,\mathrm{d}x_1\right]. \quad \text{(C 1)}$$

Here, $|_{\tilde{x}^+}$ denotes evaluation on the far downstream streamlines defined by (4.7). We now expand the term $(c_l x_l r^{-3})_{,i} = c_l(\delta_{il}r^{-3} - 3x_l x_i r^{-5})$ and write $x_l = x_1\delta_{1l} + x_{l\neq1}$ so as to distinguish between components that are parallel and perpendicular to the flow direction to obtain

$$(c_l x_l r^{-3})_{,i} = c_l\left(\delta_{il}r^{-3} - 3x_1^2 r^{-5}\delta_{1l}\delta_{1i}\right) - 3c_l\{\delta_{1l}x_{i\neq1} + \delta_{1i}x_{l\neq1}\}x_1 r^{-5} - 3c_l x_{i\neq1}x_{l\neq1}r^{-5}. \quad \text{(C 2)}$$

Writing $x_1^2 = r^2 - \rho^2$ and grouping factors of r^{-3}, $x_1 r^{-5}$ and r^{-5} we obtain

$$(c_l x_l r^{-3})_{,i} = c_l(\delta_{il} - 3\delta_{1l}\delta_{1i})r^{-3} - 3c_l\{\delta_{1l}x_{i\neq1} + \delta_{1i}x_{l\neq1}\}(x_1 r^{-5})$$
$$+ 3c_l(\rho^2\delta_{1l}\delta_{1i} - x_{i\neq1}x_{l\neq1})r^{-5}. \quad \text{(C 3)}$$

Note the following integral identities

$$\int_{x_1}^{+\infty} r^{-3}\,\mathrm{d}x_1 = \rho^{-2}[1 - (x_1 r^{-1})]; \quad \int_{x_1}^{+\infty} x_1 r^{-5}\,\mathrm{d}x_1 = \tfrac{1}{3}r^{-3}, \quad \text{(C 4}a\text{)}$$

$$\int_{x_1}^{+\infty} r^{-5}\,\mathrm{d}x_1 = \rho^{-4}\left[\tfrac{2}{3} - (x_1 r^{-1})\left\{1 - \tfrac{1}{3}(x_1 r^{-1})^2\right\}\right]. \quad \text{(C 4}b\text{)}$$

By substituting the asymptotic approximations $(x_1 r^{-1}) = x_1[\rho^2 + x_1^2]^{-1/2} \sim 1 - \rho^2/2x_1^2$, $(x_1 r^{-1})^2 \sim 1 - \rho^2/x_1^2$ and $r^{-1} = [\rho^2 + x_1^2]^{-\frac{1}{2}} \sim 1/x_1$ into (C 4), we arrive at the following approximations

$$\int_{x_1}^{+\infty} r^{-3}\,\mathrm{d}x_1 \sim \tfrac{1}{2}x_1^{-2} + O(\rho^2 x_1^{-4}); \quad \int_{x_1}^{+\infty} x_1 r^{-5}\,\mathrm{d}x_1 = \tfrac{1}{3}x_1^{-3} + O\big(\rho^2 x_1^{-5}\big), \qquad \text{(C 5}a\text{)}$$

$$\int_{x_1}^{+\infty} r^{-5}\,\mathrm{d}x_1 \sim \rho^{-4}\big[\tfrac{2}{3} - \big(1 - \tfrac{1}{2}\rho^2/x_1^2\big)\big\{\tfrac{2}{3} + \tfrac{1}{3}\rho^2/x_1^2\big\}\big] \sim \tfrac{1}{6}x_1^{-4} + O\big(\rho^2 x_1^{-6}\big). \qquad \text{(C 5}b\text{)}$$

Substituting (C5) into the integral of $(c_l x_l r^{-3})_{,i}$ we find

$$\int_{x_1}^{+\infty} (c_l x_l r^{-3})_{,i}|_{\tilde{x}^+}\,\mathrm{d}x_1 \sim \tfrac{1}{2}c_l(\delta_{il} - 3\delta_{1l}\delta_{1i})x_1^{-2} - c_l\{\delta_{1l}\lambda_i + \delta_{1i}\lambda_l\}\rho x_1^{-3} + O\big(a_B^3 \rho^2 x_1^{-4}\big). \qquad \text{(C 6)}$$

The required result now follows by differentiating with respect to $\rho^{-\infty}$ and letting $x_1 \to +\infty$, namely

$$\Delta\omega_i - \Delta\omega_i^{+\infty} \sim \Omega\lambda_3 \partial/\partial\rho^{-\infty}\left[\int_{x_1}^{+\infty} (c_l x_l r^{-3})_{,i}|_{\tilde{x}^+}\,\mathrm{d}x_1\right]$$
$$= O\big(\Omega\,\partial/\partial\rho^{-\infty}(\rho^{+\infty})a_B^3 x_1^{-3}\big) = O\big(\Omega a_B^3 x_1^{-3}\big). \qquad \text{(C 7)}$$

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